

Proof Theory and Automated Theorem Proving

Exercises

Week 1

Cory Knapp

1 Natural Deduction

$$1. \frac{\frac{\frac{[\varphi]^2}{\varphi \wedge \psi} \wedge I \quad [\psi]^1}{\varphi \wedge \psi \rightarrow \sigma} \rightarrow E \quad \frac{\sigma}{\psi \rightarrow \sigma} \rightarrow I, 1}{\frac{\psi \rightarrow \sigma}{\varphi \rightarrow (\psi \rightarrow \sigma)} \rightarrow I, 2} \rightarrow I, 3$$

$$2. \frac{\frac{[\psi \rightarrow (\varphi \rightarrow \sigma)]^2}{\varphi \rightarrow \sigma} \rightarrow E \quad \frac{\frac{[\varphi \wedge \psi]^1}{\psi} \wedge E, r \quad [\varphi \wedge \psi]^1}{\varphi} \wedge E, l}{\frac{\sigma}{\varphi \wedge \psi \rightarrow \sigma} \rightarrow I, 1} \rightarrow I, 2$$

$$3. \frac{\frac{[\varphi \wedge \chi]^1}{\varphi} \wedge E, l \quad [\varphi \rightarrow \psi]^2}{\psi} \rightarrow E \quad \frac{\psi}{\varphi \wedge \chi \rightarrow \psi} \rightarrow I, 1}{\frac{\varphi \wedge \chi \rightarrow \psi}{(\varphi \rightarrow \psi) \rightarrow (\varphi \wedge \chi \rightarrow \psi)} \rightarrow I, 2}$$

$$4. \frac{[\varphi \vee \psi]^1}{\varphi \vee (\varphi \vee \psi)} \vee I, r}{\varphi \vee \psi \rightarrow \varphi \vee (\varphi \vee \psi)} \rightarrow I, 1$$

$$5. \frac{[\varphi \vee (\varphi \vee \psi)]^3 \quad \frac{[\varphi]^1}{\varphi \vee \psi} \vee I, l \quad [\varphi \vee \psi]^2}{\varphi \vee \psi} \vee E, 1, 2}{\varphi \vee (\varphi \vee \psi) \rightarrow \varphi \vee \psi} \rightarrow I, 3$$

$$6. \frac{\frac{[\varphi]^2 \quad [\psi]^1 \wedge I}{\varphi \wedge \psi} \rightarrow I, 1}{\psi \rightarrow \varphi \wedge \psi} \rightarrow I, 2$$

$$\frac{\psi \rightarrow \varphi \wedge \psi}{\varphi \rightarrow (\psi \rightarrow \varphi \wedge \psi)} \rightarrow I, 2$$

$$7. \frac{\frac{[\varphi \wedge \psi]^1 \wedge E, r}{\psi} \quad \frac{[\varphi \wedge \psi]^1 \wedge E, l}{\varphi} \wedge I}{\psi \wedge \varphi} \rightarrow I, 1$$

$$\frac{\psi \wedge \varphi}{\varphi \wedge \psi \rightarrow \psi \wedge \varphi} \rightarrow I, 1$$

$$8. \frac{[(\varphi \wedge \psi) \vee \varphi]^3 \quad \frac{[\varphi \wedge \psi]^1 \wedge E, l}{\varphi} \quad [\varphi]^2 \vee E, 1, 2}{\varphi} \rightarrow I, 3$$

$$\frac{[(\varphi \wedge \psi) \vee \varphi]^3}{(\varphi \wedge \psi) \vee \varphi \rightarrow \varphi} \rightarrow I, 3$$

$$9. \frac{\frac{[\varphi]^1}{(\varphi \wedge \psi) \vee \varphi} \vee I, r}{\varphi \rightarrow (\varphi \wedge \psi) \vee \varphi} \rightarrow I, 1$$

$$10. \frac{\frac{[\varphi]^1 \quad \frac{[\varphi \rightarrow (\psi \wedge \chi)]^3}{\psi \wedge \chi} \wedge E, l}{\psi} \rightarrow I, 1}{\varphi \rightarrow \psi} \rightarrow E \quad \frac{[\varphi]^2 \quad \frac{[\varphi \rightarrow (\psi \wedge \chi)]^3}{\psi \wedge \chi} \wedge E, r}{\varphi \rightarrow \chi} \rightarrow I, 2}{\varphi \rightarrow \chi} \wedge I}{(\varphi \rightarrow \psi) \wedge (\varphi \rightarrow \chi)} \rightarrow I, 3$$

$$\frac{(\varphi \rightarrow \psi) \wedge (\varphi \rightarrow \chi)}{(\varphi \rightarrow (\psi \wedge \chi)) \rightarrow (\varphi \rightarrow \psi) \wedge (\varphi \rightarrow \chi)} \rightarrow I, 3$$

$$11. \frac{[\varphi]^1 \quad \frac{[(\varphi \rightarrow \psi) \wedge (\varphi \rightarrow \chi)]^2 \wedge E, l}{\varphi \rightarrow \psi} \rightarrow E}{\psi} \rightarrow E \quad \frac{[\varphi]^1 \quad \frac{[(\varphi \rightarrow \psi) \wedge (\varphi \rightarrow \chi)]^2 \wedge E, r}{\varphi \rightarrow \chi} \rightarrow E}{\chi} \wedge I}{\psi \wedge \chi} \rightarrow I, 1$$

$$\frac{\psi \wedge \chi}{\varphi \rightarrow (\psi \wedge \chi)} \rightarrow I, 1$$

$$\frac{\varphi \rightarrow (\psi \wedge \chi)}{(\varphi \rightarrow \psi) \wedge (\varphi \rightarrow \chi) \rightarrow (\varphi \rightarrow (\psi \wedge \chi))} \rightarrow I, 2$$

$$12. \frac{\frac{[\varphi]^1}{\varphi \vee \psi} \vee I, l \quad \frac{[\varphi \vee \psi \rightarrow \chi]^3}{\chi} \rightarrow I, 1}{\varphi \rightarrow \chi} \rightarrow I, 1 \quad \frac{\frac{[\psi]^2}{\varphi \vee \psi} \vee I, r \quad \frac{[\varphi \vee \psi \rightarrow \chi]^3}{\chi} \rightarrow I, 2}{\psi \rightarrow \chi} \rightarrow I, 2}{\psi \rightarrow \chi} \wedge I}{(\varphi \rightarrow \chi) \wedge (\psi \rightarrow \chi)} \rightarrow I, 3$$

$$\frac{(\varphi \rightarrow \chi) \wedge (\psi \rightarrow \chi)}{(\varphi \vee \psi \rightarrow \chi) \rightarrow ((\varphi \rightarrow \chi) \wedge (\psi \rightarrow \chi))} \rightarrow I, 3$$

$$\begin{array}{l}
13. \quad \frac{\frac{[(\varphi \vee \psi) \wedge (\psi \rightarrow \varphi)]^3}{\varphi \vee \psi} \wedge E, l \quad [\varphi]^1 \quad \frac{[\psi]^2 \quad \frac{[(\varphi \vee \psi) \wedge (\psi \rightarrow \varphi)]^3}{\psi \rightarrow \varphi} \wedge E, r}{\varphi} \rightarrow E}{\varphi} \vee E, 1, 2}{\frac{\varphi}{(\varphi \vee \psi) \wedge (\psi \rightarrow \varphi)} \rightarrow \varphi} \rightarrow I, 3 \\
\\
14. \quad \frac{[\varphi \vee \psi]^3 \quad \frac{[\varphi]^1 \quad \frac{[(\varphi \rightarrow \chi) \vee (\psi \rightarrow \chi)]^4}{\varphi \rightarrow \chi} \wedge E, l}{\chi} \rightarrow E \quad \frac{[\psi]^2 \quad \frac{[(\varphi \rightarrow \chi) \vee (\psi \rightarrow \chi)]^4}{\psi \rightarrow \chi} \wedge E, r}{\chi} \rightarrow E}{\chi} \vee E, 1, 2}{\frac{\chi}{\varphi \vee \psi \rightarrow \chi} \rightarrow I, 3} \rightarrow I, 4}{\frac{\varphi \vee \psi \rightarrow \chi}{(\varphi \rightarrow \chi) \vee (\psi \rightarrow \chi) \rightarrow (\varphi \vee \psi \rightarrow \chi)} \rightarrow I, 4} \\
\\
15. \quad \frac{\frac{[\psi]^1}{\varphi \vee \psi} \vee I, r \quad \frac{[\psi]^1}{\neg \varphi \vee \psi} \vee I, r}{\frac{(\varphi \vee \psi) \wedge \neg \varphi \vee \psi}{\psi \rightarrow (\varphi \vee \psi) \wedge \neg \varphi \vee \psi} \wedge I} \rightarrow I, 1 \\
\\
16. \quad \frac{\frac{[(\varphi \vee \psi) \wedge (\neg \varphi \vee \psi)]^5}{\neg \varphi \vee \psi} \wedge E, r \quad \frac{[\varphi]^3 \quad [\neg \varphi]^1}{\perp} \rightarrow E \quad \frac{[\psi]^2}{\psi} \vee E, 1, 2}{\frac{[\psi]^4}{\psi} \vee E, 3, 4} \rightarrow I, 5}{\frac{[(\varphi \vee \psi) \wedge (\neg \varphi \vee \psi)]^5}{\varphi \vee \psi} \wedge E, l \quad \frac{[\psi]^4}{\psi} \vee E, 3, 4}{\frac{\psi}{(\varphi \vee \psi) \wedge (\neg \varphi \vee \psi)} \rightarrow \psi} \rightarrow I, 5} \\
\\
17. \quad \frac{[(\varphi \wedge \psi) \vee (\neg \varphi \wedge \psi)]^3 \quad \frac{[\varphi \wedge \psi]^1}{\psi} \wedge E, r \quad \frac{[\neg \varphi \wedge \psi]^2}{\psi} \wedge E, r}{\psi} \vee E, 1, 2}{\frac{\psi}{(\varphi \wedge \psi) \vee (\neg \varphi \wedge \psi)} \rightarrow \psi} \rightarrow I, 3 \\
\\
18. \quad \frac{\mathcal{D} \quad \frac{[\varphi]^3 \quad [\psi]^5}{\varphi \wedge \psi} \wedge I \quad \frac{[\neg \varphi]^4 \quad [\psi]^5}{\varphi \wedge \psi} \wedge I}{\frac{\varphi \wedge \psi}{(\varphi \wedge \psi) \vee (\neg \varphi \wedge \psi)} \vee I, l \quad \frac{\varphi \wedge \psi}{(\varphi \wedge \psi) \vee (\neg \varphi \wedge \psi)} \vee I, r} \vee E, 3, 4}{\frac{(\varphi \wedge \psi) \vee (\neg \varphi \wedge \psi)}{\psi \rightarrow (\varphi \wedge \psi) \vee (\neg \varphi \wedge \psi)} \rightarrow I, 5} \rightarrow I, 5}
\end{array}$$

Where \mathcal{D} is the proof of $\varphi \vee \neg \varphi$. Namely,

$$\frac{\frac{[\varphi]^1}{\varphi \vee \neg\varphi} \vee I, l \quad [\neg(\varphi \vee \neg\varphi)]^2}{\rightarrow E} \rightarrow E$$

$$\frac{\frac{\perp}{\neg\varphi} \rightarrow I, 1}{\varphi \vee \neg\varphi} \vee I, r \quad \frac{[\neg(\varphi \vee \neg\varphi)]^2}{\rightarrow E} \rightarrow E}{\frac{\perp}{\varphi \vee \neg\varphi} \text{RAA}, 2} \rightarrow E$$

$$19. \quad \frac{\frac{[\neg\neg\varphi]^2}{\varphi} \text{RAA}, 1 \quad \frac{[\neg\varphi]^1}{\neg\neg\varphi \rightarrow \varphi} \rightarrow I, 2}{\rightarrow E} \rightarrow E$$

$$20. \quad \frac{\frac{[\varphi]^1}{\varphi \rightarrow \psi} \rightarrow I, 1 \quad \frac{[\neg\varphi]^2}{\psi} \perp}{\varphi} \rightarrow E \quad \frac{[(\varphi \rightarrow \psi) \rightarrow \varphi]^3}{\varphi} \rightarrow E \quad \frac{[\neg\varphi]^2}{\rightarrow E} \rightarrow E}{\frac{\perp}{\varphi} \text{RAA}, 2} \rightarrow E \quad \frac{((\varphi \rightarrow \psi) \rightarrow \varphi) \rightarrow \varphi}{\rightarrow I, 3} \rightarrow E$$

2 Constructions on proofs

1. Let $\frac{\mathcal{D}}{\varphi \wedge \psi}$ be a proof of $\varphi \wedge \psi$. Then $\frac{\mathcal{D}}{\varphi} \wedge E, l$ is a proof of φ , and $\frac{\mathcal{D}}{\psi} \wedge E, r$ is a proof of ψ .
2. Let $\frac{\mathcal{D}}{\varphi \vee \psi}$ be a proof of $\varphi \vee \psi$ and $\frac{\mathcal{D}'}{\neg\varphi}$ be a proof of $\neg\varphi$. Then

$$\frac{\frac{\mathcal{D}}{\varphi \vee \psi} \quad \frac{[\varphi]^1 \quad \frac{\mathcal{D}'}{\neg\varphi} \rightarrow E}{\perp} \perp}{\psi} \vee E, 1, 2$$

is a proof of ψ .

3. Let $\frac{\mathcal{D}}{\varphi}$ be a proof of φ and $\frac{\mathcal{D}'}{\psi}$ be a proof of ψ . Then $\frac{\frac{\mathcal{D}}{\varphi} \quad \frac{\mathcal{D}'}{\psi}}{\varphi \wedge \psi} \wedge I$ is a proof of $\varphi \wedge \psi$.
4. Let $\frac{\mathcal{D}}{\varphi \rightarrow \psi}$ be a proof of $\varphi \rightarrow \psi$ and $\frac{\mathcal{D}'}{\neg\psi}$ be a proof of $\neg\psi$. Then

$$\frac{\frac{[\varphi]^1 \quad \varphi \rightarrow \psi \xrightarrow{\mathcal{D}} \rightarrow E \quad \mathcal{D}' \quad \neg\psi}{\psi} \rightarrow E}{\frac{\perp}{\neg\varphi} \rightarrow I, 1} \rightarrow E$$

is a proof of $\neg\varphi$.

5. Let $\varphi \rightarrow \neg\varphi \xrightarrow{\mathcal{D}}$ be a proof of $\varphi \rightarrow \neg\varphi$. Then

$$\frac{[\varphi]^1 \quad \frac{[\varphi]^1 \quad \varphi \rightarrow \neg\varphi \xrightarrow{\mathcal{D}} \rightarrow E}{\neg\varphi} \rightarrow E}{\frac{\perp}{\neg\varphi} \rightarrow I, 1} \rightarrow E$$

is a proof of $\neg\varphi$.

6. Let $\varphi \rightarrow \neg\varphi \xrightarrow{\mathcal{D}}$ be a proof of $\varphi \rightarrow \neg\varphi$. By 5, we have a proof \mathcal{D}' of $\neg\varphi$. Let $\neg\varphi \xrightarrow{\mathcal{E}} \varphi$ be a proof of $\neg\varphi \rightarrow \varphi$. Then

$$\frac{\frac{\mathcal{D}' \quad \neg\varphi \quad \neg\varphi \rightarrow \varphi \xrightarrow{\mathcal{E}} \rightarrow E \quad \mathcal{D}' \quad \neg\varphi}{\varphi} \rightarrow E}{\perp} \rightarrow E$$

is a proof of \perp .

3 Intuitionistic logic

$$1. \quad \frac{[\varphi]^2 \quad [\neg\varphi]^1 \rightarrow E}{\frac{\perp}{\neg\neg\varphi} \rightarrow I, 1} \rightarrow I, 2$$

$$2. \quad \frac{\frac{[\varphi]^1 \quad \varphi \rightarrow \neg\neg\varphi \xrightarrow{\mathcal{D}} \rightarrow E}{\frac{\perp}{\neg\neg\varphi} \rightarrow E} \quad [\neg\neg\varphi]^2 \rightarrow E}{\frac{\perp}{\neg\neg\varphi} \rightarrow I, 1} \rightarrow I, 2$$

Where \mathcal{D} is the proof of $\varphi \rightarrow \neg\neg\varphi$ above.

3. (with help from Esperanza Buitrago Díaz)

$$\frac{\frac{[\varphi]^1 \quad [\psi]^2}{\varphi \wedge \psi} \wedge I \quad [\neg(\varphi \wedge \psi)]^3 \rightarrow E}{\frac{\perp}{\neg\varphi} \rightarrow I, 1} \rightarrow E \quad \frac{[\neg\varphi \wedge \neg\neg\psi]^4}{\neg\neg\varphi} \wedge E, l \rightarrow E}{\frac{\perp}{\neg\psi} \rightarrow I, 2 \quad \frac{[\neg\varphi \wedge \neg\neg\psi]^4}{\neg\neg\psi} \wedge E, r \rightarrow E} \rightarrow E$$

$$\frac{\perp}{\neg\neg(\varphi \wedge \psi)} \rightarrow I, 3$$

$$\frac{\perp}{\neg\neg\varphi \wedge \neg\neg\psi \rightarrow \neg\neg(\varphi \wedge \psi)} \rightarrow I, 4$$