Proof Theory and Automated Theorem Proving
Exercises
Week 2

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1 Terms, Forumulas, and Free and Bound Variables

For a signature S with constants $\mathscr{C}_S$, functions $\mathscr{F}_S$ and relations $\mathscr{R}_S$, the set $\mathrm{TERM}(S)$ of S-terms is the least class T such that

- if x is a variable, then $x \in T$,
- if $c \in \mathscr{C}_S$, then $c \in T$,
- if $f \in \mathscr{F}_S$ is n-ary and $t_1, \ldots, t_n \in T$, then $f(t_1, \ldots, t_n) \in T$.

The set $\mathrm{FORM}(S)$ of S-formulas is the least class Σ such that

- $\bot \in \Sigma$,
- if $R \in \mathscr{R}_S$ is n-ary and $t_1, \ldots, t_n \in \mathrm{TERM}(S)$, then $R(t_1, \ldots, t_n) \in \Sigma$,
- if $\varphi, \psi \in \Sigma$, then $(\varphi \to \psi) \in \Sigma$, $(\varphi \wedge \psi) \in \Sigma$ and $(\varphi \vee \psi) \in \Sigma$,
- if x is a variable and $\varphi \in \Sigma$ then $\forall x \varphi \in \Sigma$ and $\exists x \varphi \in \Sigma$.

For a term t the set $\mathrm{FV}(t)$ is defined inductively by

- $\mathrm{FV}(x) = \{x\}$ for a variable x,
- $\mathrm{FV}(f(t_1, \ldots, t_n)) = \bigcup_{1 \leq i \leq n} \mathrm{FV}(t_i)$, when $f \in \mathscr{F}_S$ is n-ary and $t_1, \ldots, t_n \in \mathrm{TERM}(S)$,
- $\mathrm{FV}(c) = \emptyset$ for a constant $c \in \mathscr{C}_S$,

For a formula φ, the set $\mathrm{FV}(\varphi)$ of free variables is defined inductively by

- $\mathrm{FV}(\bot) = \emptyset$,
- $\mathrm{FV}(R(t_1, \ldots, t_n)) = \bigcup_{1 \leq i \leq n} \mathrm{FV}(t_i)$, where $R \in \mathscr{R}_S$ is n-ary and $t_1, \ldots, t_n \in \mathrm{TERM}(S)$,
- if $\varphi = \psi \to \chi$, $\varphi = \psi \vee \chi$ or $\varphi = \psi \wedge \chi$ for $\psi, \chi \in \mathrm{FORM}(S)$, then $\mathrm{FV}(\varphi) = \mathrm{FV}(\psi) \cup \mathrm{FV}(\chi)$,
- if $\varphi = \forall x \psi$ or $\varphi = \exists x \psi$, where $\psi \in \mathrm{FORM}(S)$ and x is a variable, then $\mathrm{FV}(\varphi) = \mathrm{FV}(\psi) \setminus \{x\}$.

Similarly, the set $\mathrm{BV}(\varphi)$ of bound variables is defined inductively by

- $\mathrm{BV}(\bot) = \emptyset$,

- $\mathrm{BV}(R(t_1,\dots,t_n)) = \emptyset$, where $R \in \mathscr{R}_S$ is n-ary and $t_1,\dots,t_n \in \mathrm{TERM}(S)$,
- if $\varphi = \psi \to \chi$, $\varphi = \psi \vee \chi$ or $\varphi = \psi \wedge \chi$ for $\psi, \chi \in \mathrm{FORM}(S)$, then $\mathrm{BV}(\varphi) = \mathrm{BV}(\psi) \cup \mathrm{BV}(\chi)$,
- if $\varphi = \forall x \psi$ or $\varphi = \exists x \psi$, where $\psi \in \mathrm{FORM}(S)$ and x is a variable, then $\mathrm{BV}(\varphi) = \mathrm{BV}(\psi) \cup \{x\}$.

A variable x is free or bound in a formula φ if $x \in \mathrm{FV}(\varphi)$ or $x \in \mathrm{BV}(\varphi)$, respectively. Similarly, x is free in a term t if $x \in \mathrm{FV}(T)$.

For a term t and a variable x, t is free for x in φ is defined inductively on formulas by

- If $\varphi = \bot$, then t is free for x in φ.
- If $\varphi = R(t_1,\dots,t_n)$ for an n-ary relation R and terms $t_1,\dots,t_n$, then t is free for x in φ.
- If $\varphi = \psi \to \chi$, $\varphi = \psi \vee \chi$ or $\varphi = \psi \wedge \chi$ for $\psi, \chi \in \mathrm{FORM}(S)$, then t is free for x in φ if and only if t is free for x in ψ and t is free for x in χ.
- If $\varphi = \forall y \psi$, then t is free for x in φ if and only if the following conditions hold
  - $x \neq y$.
  - $y \notin \mathrm{FV}(t)$
  - t is free for x in ψ.

2 Predicate Logic and Natural Deduction

$$\dfrac{\dfrac{\dfrac{\dfrac{[\forall x \varphi(x)]^2}{\varphi(x)}\,\forall E \qquad \dfrac{[\forall x \neg\varphi(x)]^1}{\neg\varphi(x)}\,\forall E}{\bot}\,\to E}{\neg\forall x \neg\varphi(x)}\,\to I, 1}{\forall x \varphi(x) \to \neg\forall x \neg\varphi(x)}\,\to I, 2$$