Proof Theory and Automated Theorem Proving
Exercises
Week 2

Cory Knapp

4 Tait Language

Exercise For every formula F in a first order language $\mathscr{L}$, there is a formula F^T in the Tait language $\mathscr{L}_T$.

1. For a formula F in a first order language $\mathscr{L}$, define F^T a formula in $\mathscr{L}_F$ inductively as follows.
   - If F is atomic, $F^T = F$;
   - if $F = A \vee B$ for $\mathscr{L}$-formulas A, B then $F^T = A^T \vee B^T$;
   - if $F = A \wedge B$ for $\mathscr{L}$-formulas A, B then $F^T = A^T \wedge B^T$;
   - if $F = A \to B$ for $\mathscr{L}$-formulas A, B then $F^T = (\neg A \vee B)^T$;
   - if $F = \neg A^T$ for an $\mathscr{L}$-formula A, then $F^T = \sim (A^T)$;
   - If $F = \forall x A_u(x)$, for an $\mathscr{L}$-formula A, and a variable x, then $F^T = \forall x(A_u^T(x))$;
   - If $F = \exists x A_u(x)$, for an $\mathscr{L}$-formula A, and a variable x, then $F^T = \exists x(A_u^T(x_i))$,
2. For an $\mathscr{S}$ structure $\mathbb{M}$ and an $\mathscr{L}$-formula F, $\mathbb{M} \models F$ iff the corresponding $\mathscr{L}_T$ structure $\mathbb{M}_T$ satisfies $\mathbb{M} \models F^T$. We show this inductively.
   - If F is atomic, then F and F^T, have the same interpretation.
   - If $F = A \vee B$, then $\mathbb{M} \models F$ if and only if $\mathbb{M} \models A$ or $\mathbb{M} \models B$. By the inductive hypothesis, this is equivalent to $\mathbb{M}_T \models A^T$ or $\mathbb{M}_T \models B^T$. Hence $\mathbb{M} \models F$ if and only if $\mathbb{M}_T \models A^T \vee B^T$. But $\mathbb{M}_T \models A^T \vee B^T \Leftrightarrow \mathbb{M}_T \models (A \vee B)^T$
   - Similarly, if $F = A \wedge B$, then $\mathbb{M} \models F$ precisely when $\mathbb{M} \models A$ and $\mathbb{M} \models B$, hence when $\mathbb{M}_T \models A^T$ and $\mathbb{M}_T \models B^T$. Thus $\mathbb{M} \models A \wedge B \Leftrightarrow \mathbb{M}_t \models (A \wedge B)^T$.
   - If $F = \neg A$, then $\mathbb{M} \models F$ precisely if $\mathbb{M} \not\models A$. By the inductive hypothesis, this is equivalent to $\mathbb{M}_T \not\models A^T$, which happens precisely when $\mathbb{M}_T \models \sim A^T$, again giving the desired equivalence.
   - Since $A \to B$ is classically equivalent to $\neg A \vee B$, the implicative case reduces to the disjunctive case.
   - If $F = \forall u A(u)$, then $\mathbb{M} \models F$ precisely when $\mathbb{M} \models A(m)$ for all $m \in M$. But in this case $\mathbb{M}_T \models A(m)^T$ for all $m \in M$. Hence, $\mathbb{M} \models F \Leftrightarrow \mathbb{M}_T \models F^T$.
   - Similarly, $\mathbb{M} \models \exists u A(u)$ precisely when there is an $m \in M$ such that $\mathbb{M} \models A(m)$. This happens precisely when $\mathbb{M}_T \models A(m)^T$. Thus $\mathbb{M} \models \exists x A(x) \Leftrightarrow \mathbb{M}_T \models (\exists x A(x))^T$.

5 Predicate logic in Tait calculus

1. $(\exists x(\varphi(x) \to \psi) \to (\forall x \varphi(x) \to \psi))^T = \forall x(\varphi(x) \wedge \neg\psi) \vee (\exists x \neg\varphi(x) \vee \psi)$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\neg\varphi(u),\psi,\varphi(u)}{\exists x\neg\varphi(x),\psi,\varphi(u)}\,(\exists)\qquad \exists x\neg\varphi(x),\psi,\neg\psi}{\varphi(u)\wedge\neg\psi,\exists x\neg\varphi(x),\psi}\,(\wedge)}{\forall x(\varphi(x)\wedge\neg\psi),\exists x\neg\varphi(x),\psi}\,(\forall)}{\forall x(\varphi(x)\wedge\neg\psi)\vee(\exists x\neg\varphi(x)\vee\psi),\exists x\neg\varphi(x),\psi}\,(\vee)}{\forall x(\varphi(x)\wedge\neg\psi)\vee(\exists x\neg\varphi(x)\vee\psi),\exists x\neg\varphi(x)\vee\psi,\psi}\,(\vee)}{\forall x(\varphi(x)\wedge\neg\psi)\vee(\exists x\neg\varphi(x)\vee\psi),\exists x\neg\varphi(x)\vee\psi}\,(\vee)}{\forall x(\varphi(x)\wedge\neg\psi)\vee(\exists x\neg\varphi(x)\vee\psi)}\,(\vee)$$

2. $((\forall x\varphi(x)\to\psi)\to\exists x(\varphi(x)\to\psi))^T=(\forall x\varphi(x)\wedge\neg\psi)\vee\exists x(\neg\varphi(x)\vee\psi)$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\varphi(u),\neg\varphi(u)}{\varphi(u),\neg\varphi(u)\vee\psi}\,(\vee)}{\varphi(u),\exists x(\neg\varphi(x)\vee\psi)}\,(\exists)}{\forall x\varphi(x),\exists x(\neg\varphi(x)\vee\psi)}\,(\forall)\qquad \dfrac{\dfrac{\neg\psi,\psi}{\neg\psi,\neg\varphi(u)\vee\psi}\,(\vee)}{\neg\psi,\exists x(\neg\varphi(x)\vee\psi)}\,(\exists)}{\forall x\varphi(x)\wedge\neg\psi,\exists x(\neg\varphi(x)\vee\psi)}\,(\wedge)}{(\forall x\varphi(x)\wedge\neg\psi)\vee\exists x(\neg\varphi(x)\vee\psi),\exists x(\neg\varphi(x)\vee\psi)}\,(\vee)}{(\forall x\varphi(x)\wedge\neg\psi)\vee\exists x(\neg\varphi(x)\vee\psi)}\,(\vee)$$

3. $(\forall x(\varphi(x)\to\psi(x))\to(\forall x\varphi(x)\to\forall x\psi(x)))^T=\exists x(\varphi(x)\wedge\neg\psi(x))\vee(\exists x\neg\varphi(x)\vee\forall x\psi(x))$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\varphi(u),\neg\varphi(u),\psi(u)\qquad \neg\psi(u),\neg\varphi(u),\psi(u)}{\varphi(u)\wedge\neg\psi(u),\neg\varphi(u),\psi(u)}\,(\wedge)}{\exists x(\varphi(x)\wedge\neg\psi(x)),\neg\varphi(u),\psi(u)}\,(\exists)}{\exists x(\varphi(x)\wedge\neg\psi(x)),\exists(x)\neg\varphi(x),\psi(u)}\,(\exists)}{\exists x(\varphi(x)\wedge\neg\psi(x)),\exists(x)\neg\varphi(x),\forall x\psi(x)}\,(\forall)}{\exists x(\varphi(x)\wedge\neg\psi(x))\vee(\exists(x)\neg\varphi(x)\vee\forall x\psi(x)),\exists(x)\neg\varphi(x),\forall x\psi(x)}\,(\vee)}{\exists x(\varphi(x)\wedge\neg\psi(x))\vee(\exists(x)\neg\varphi(x)\vee\forall x\psi(x)),\exists(x)\neg\varphi(x)\wedge\forall x\psi(x),\forall x\psi(x)}\,(\vee)}{\exists x(\varphi(x)\wedge\neg\psi(x))\vee(\exists(x)\neg\varphi(x)\vee\forall x\psi(x)),\exists(x)\neg\varphi(x)\wedge\forall x\psi(x)}\,(\vee)}{\exists x(\varphi(x)\wedge\neg\psi(x))\vee(\exists(x)\neg\varphi(x)\vee\forall x\psi(x))}\,(\vee)$$