Proof Theory and Automated Theorem Proving
Exercises
Week 3b

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1 Primitive recursive well-orders

1. The natural ordering $<$ on $\mathbb{N}$ has order type ω, so it suffices to show that $<$ is primitive recursive. Let $(a,b) \in \mathbb{N}^2$. Then $a < b$ if and only if counting down from b we eventually reach a. More precisely, let h be the function

$$h(n,m,t) = \begin{cases} 1 & \text{if } n = m \\ t & \text{otherwise} \end{cases}$$

   Then $\chi_<(a,b) = Rec(h, c_0)(b,a)$, so $<$ is a primitive recursive relation.

2. Let $A = \{p^k : k \in \mathbb{N},\ p \text{ prime}\}$. Define $\prec$ on A by $p^k \prec q^k$ if and only if $p < q$ or $p = q$ and $k < l$. This is order isomorphic to ω^2 by sending p_i^k to $\omega \cdot i + k$, where p_i is the i-th prime.

   Prime factorization and comparing natural numbers are primitive recursive, so this relation is.

3. Let $\alpha < \omega^\omega$ be an ordinal. We can express α as $\omega^n \cdot c_n + \ldots + \omega \cdot c_1 + c_0$ where $c_0, \ldots, c_n \in \mathbb{N}$ and $c_n > 0$. The important information is contained in the sequence $\langle c_n, \ldots, c_0 \rangle$. In other words, there is a natural bijection between ω^ω and the set S of finite sequences on $\mathbb{N}$ whose first element is non-zero given by $\alpha \mapsto \langle c_n, \ldots, c_0 \rangle$.

   Moreover, this bijection induces an ordering on S given by $\langle c_n, \ldots, c_0 \rangle \preceq \langle d_m, \ldots, d_0 \rangle$ if and only if

   - $n < m$,
   - $n = m$ and there is some $k \leq n$ such that $c_i = d_i$ for $i > k$ and $c_k < d_k$ or
   - $n = m$ and $c_i = d_i$ for all $i \leq n$.

   Checking whether a number is a sequence number, finding the length of a sequence and checking if the first coordinate of a sequence is non-zero are all primitive recursive operations. Since the natural ordering on $\mathbb{N}$ is also primitive recursive, the ordering $\preceq$ is primitive recursive.

2 Primitive recursive trees

(a)

Claim. *A tree T is well-founded if and only if the tree-ordering $\prec_T$ is well-founded.*

Notice first that for $s, t \in T$, $s \prec t$ if and only if $t \subsetneq s$.

Suppose T is not well-founded, and let P be an infinite path in T. Then $P = \{t_i : i \in \mathbb{N}\}$, and $t_i \subsetneq t_j$ if $i \leq j$. Hence, $t_i \succ t_j$ if $i \leq j$. But then P has no $\prec$-least element, so $\prec$ is not well-founded.

Conversely, suppose $\prec$ is not well-founded, and let $S \subset T$ be an infinite descending $\prec$-chain. Consider S^* the closure of S under initial segments. S^* is an infinite path in T, so T is not well-founded.

(b)

Claim. *For any tree T,* $\mathrm{depth}(T) = \mathrm{otyp}(T)$*.*

We show first that $\mathrm{depth}(T) = \sup\{\mathrm{depth}(T_{\langle a\rangle}) + 1 : \langle a\rangle \in T\}$ by induction on the order type of T. We will call a function $f : T \to \alpha$ from T to an ordinal *deep* if f is such that $s \prec_T t \Rightarrow f(s) < f(t)$.

Suppose $T = \{\langle\rangle\}$. That is, $\mathrm{otyp}(T) = 0$. Then the least ordinal α such that there is a function from T to α is 1, and any function from $\{\langle\rangle\}$ to an ordinal is deep.

Now suppose $\mathrm{otyp}(T) > 0$. Since $\mathrm{otyp}(T_{\langle a\rangle}) < \mathrm{otyp}(T)$ for each $\langle a\rangle \in T$, by the inductive hypothesis there is a deep function $f_a : T_{\langle a\rangle} \to \mathrm{depth}(T_{\langle a\rangle}) + 1$. Let $\alpha = \sup\{\mathrm{depth}(T_{\langle a\rangle}) + 1 : \langle a\rangle \in T\}$ and define $f : T \to \alpha + 1$ by $f(\langle\rangle) = \alpha$ and $f(\langle a\rangle \frown s) = f_a(s)$.

f is surely deep. Suppose there is $\beta < \alpha$ such that there is a deep function g from T to $\beta + 1$. Then there is $\langle a\rangle \in T$ such that $\beta < \mathrm{depth}(T_{\langle a\rangle})$. Define $g_a : T_{\langle a\rangle} \to \beta + 1$ by $g_a(s) = g(\langle a\rangle \frown s)$. Then g is a deep function from $T_{\langle a\rangle}$ to $\beta + 1$, but $\mathrm{depth}(T_{\langle a\rangle})$ is the least ordinal such that there is a deep function $T_{\langle a\rangle} \to \mathrm{depth}(T_{\langle a\rangle})$, so this cannot happen. So $\mathrm{depth}(T) = \sup\{\mathrm{depth}(T_{\langle a\rangle}) + 1 : \langle a\rangle \in T\}$

Now by induction on $\mathrm{otyp}(T)$ we show that $\mathrm{otyp}(T) = \mathrm{depth}(T)$ for all trees T.

- $\mathrm{otyp}(\{\langle\rangle\} = \mathrm{depth}(\{\langle\rangle\}) = 0$;
- If $\mathrm{otyp}(T) > 0$, then $\mathrm{otyp}(T) = \sup\{\mathrm{otyp}(T_{\langle a\rangle}) : \langle a\rangle \in T\}$. By the inductive hypothesis,

  $$\sup\{\mathrm{otyp}(T_{\langle a\rangle}) : \langle a\rangle \in T\} = \sup\{\mathrm{depth}(T_{\langle a\rangle}) : \langle a\rangle \in T\},$$

  so $\mathrm{otyp}(T) = \mathrm{depth}(T)$.

(c)

Claim. *For a tree T, $<_{T_{KB}}$ is a linear ordering.*

- For $s \in T$, $s \not\prec_T s$ and $(s)_i = (s)_i$ for all $i < lh(s)$, so $<_{T_{KB}}$ is irreflexive.
- Let $s, t, u \in T$ such that $s <_{T_{KB}} t$ and $t <_{T_{KB}} u$. There are two cases for $s <_{T_{KB}} t$;
  - $s \prec_T t$: If $t \prec_T u$, then $s \prec_T u$. If instead there is some $b \in T$ such that $b \subseteq t$, $b \subseteq u$ and $(t)_{lh(b)} < (u)_{lh(b)}$. If $lh(b) > lh(s)$, then $s \prec_T u$. If not, then $(s)_{lh(b)} = (t)_{lh(b)} < (u)_{lh(b)}$. In either case, $s <_{T_{KB}} u$;

- there is some b such that $b \subseteq t, s$ and $(s)_{lh(b)} < (t)_{lh(b)}$. If $t \prec_T u$, then the same b witnesses that $s <_{T_{KB}} u$. If there is some $c \subseteq t, u$ such that $(t)_{lh(c)} < (u)_{lh(c)}$, then $a = \min_{\subseteq}\{b, c\}$ is such that $(s)_{lh(a)} < (u)_{lh(a)}$ In either case, $s <_{T_{KB}} u$.

- Let $s, t \in T$ such that $s \neq t$. There are 2 cases:
  - $s \prec_T t$ or $t \prec_T s$; without loss of generality, $s \prec_T t$. Then $s <_{T_{KB}} t$;
  - Neither $s \prec t$ nor $t \prec s$. As T is a tree, there is a longest $b \in T$ such that $b \subseteq s$ and $b \subseteq t$. Since b is the longest such, $(s)_{lh(b)} \neq (t)_{lh(b)}$. Hence, $(s)_{lh(b)} < (t)_{lh(b)}$ or $(t)_{lh(b)} < (s)_{lh(b)}$. So $<_{T_{KB}}$ is total.

That is, $<_{T_{KB}}$ is a linear order.

(d)

Claim. *A tree T is well-founded if and only if $<_{T_{KB}}$ is a well-founded relation on T.*

Suppose that P is an infinite path on T. Then P is an infinite descending $<_{T_{KB}}$-chain.

For the converse, suppose towards a contradiction that T is well-founded and S is an infinite descending $<_{T_{KB}}$-chain. Since there is no infinite path on T, there must be an infinite sub-chain C of S such that no two elements of C are comparable with $\subseteq$. Index C as $\{c_i : i \in \mathbb{N}\}$ such that $c_i >_{T_{KB}} c_j$ when $i < j$. Inductively define b_i for $i \in \mathbb{N}$ as follows:

- $b_0 = \langle\rangle$;
- b_{i+1} is shortest such that $b_i \subsetneq b_{i+1}$ and $b_{i+1} \subseteq c_i, c_{i+1}$ if such an element exists. Otherwise, let $b_{i+1} = b_i$.

Notice that $b_{i+1} = b_i$ only when $(c_i)_{lh(b_i)} > (c_{i+1})_{lh(b_i)}$; otherwise, $b_i \frown \langle (c_i)_{lh(b_i)} \rangle$ satisfies the desired conditions. $\{b_i : i \in \mathbb{N}\}$ is a path in T. Since it cannot be infinite, there must be some k such that for all $i > k$, $b_i = b_{i+1}$. But then for $i > k$, it follows that $(c_i)_{lh(b_i)} > (c_{i+1})_{lh(b_i)}$. But this is an infinite descending chain in $\mathbb{N}$, a contradiction.

(e)

Claim. $\mathrm{otyp}(<_{T_{KB}}) \leq \omega^{\mathrm{depth}(T)} + \mathrm{depth}(T)$

By induction on $\mathrm{depth}(T)$:

- if $\mathrm{depth}(T) = 0$, then $\mathrm{otyp}(<_{T_{KB}}) = 1$, as $|T| = 1$;
- if $\mathrm{depth}(T) = \alpha + 1$ is a successor ordinal, then

$$\begin{aligned}\mathrm{otyp}(<_{T_{KB}}) &= \sup_{\langle a\rangle \in T} (\mathrm{otyp}(<_{T_{\langle a\rangle KB}})) + 1 \\ &\leq (\omega^\alpha + \alpha) \cdot \omega + 1 \\ &= \omega^{\mathrm{depth}(T)} + \mathrm{depth}(T).\end{aligned}$$

  The inequality above follows from the inductive hypothesis, and the fact that $\mathrm{depth}(T_{\langle a\rangle}) < \alpha$;

- if $\mathrm{depth}(T) = \alpha$ is a limit ordinal, then

$$\begin{aligned}\mathrm{otyp}(<_{T_{KB}}) &= \sup_{\langle a\rangle \in T} (\mathrm{otyp}(<_{T_{\langle a\rangle KB}})) + 1 \\ &\leq \sup(\omega^{\mathrm{depth}(T_{\langle a\rangle})} + \mathrm{depth}(T_{\langle a\rangle})) + 1 \\ &\leq \omega^{\mathrm{depth}(T)} + \mathrm{depth}(T).\end{aligned}$$

  Again, the first inequality follows from the inductive hypothesis.

(f)

Claim. *For any primitive recursive tree T,* $\mathrm{otyp}(T) < \omega_1^{CK}$.

As T is primitive recursive, the Kleene-Brouwer order on T is. So we have

$$\mathrm{otyp}(T) = \mathrm{depth}(T) \leq \mathrm{otyp}(<_{T_{KB}}) < \omega_1^{CK}.$$