Proof Theory an Automated Theorem Proving 2013
Midterm exam

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April 25, 2013

1 Proof-depth

In this exercise, let $x \mid y$ denote the primitive recursive relation that x divides y, i.e., there is some natural number m so that $x \cdot m = y$. Moreover, for two natural numbers n, m let us denote by (m, n) the primitive recursive function that yields the *greatest common divisor* of m and n. (That is, the largest number that divides both m and n.) In this exercise, p and p' range over prime numbers. We enumerate the primes by $p_1 = 2, p_2 = 3, p_3 = 5, p_4 = 7, \ldots$ and note that p_0 is undefined. We define the *index* function ind on the primes as $\mathsf{ind}(p_i) := i$. Thus, e.g., $\mathsf{ind}(7) = 4$. We consider the following primitive recursive relation $\prec$ which is recursively defined as

$$n \prec m \quad := \quad n \neq m \wedge \Big[n = 1 \vee m = 0 \vee \big(\forall\, p \mid \tfrac{n}{(m,n)}\big)\big(\exists\, p' \mid \tfrac{m}{(m,n)}\big)\ \mathsf{ind}(p) \prec \mathsf{ind}(p') \Big]$$

The order type of this relation is ε_0.

1. Show that $m \mid n$ implies $m \preceq n$.
2. Show that $7 \prec 5$.
3. Show that $\prec$ is well-defined and that it is transitive and irreflexive.
4. The ordinal $\underline{1}$ is represented in $\prec$ by 2. Find a natural number which represents ω in $\prec$ and prove that its ordertype is indeed ω.
5. Prove that the ordertype of 5 is ω^ω.
6. Find the unique number whose ordertype is ω^{ω^ω}.
7. Argue to the extend that for any number $m > 0$ and any arithmetic formula F we have that

   $$\mathsf{PA} \vdash \forall x (\forall\, y{\prec}x\ F(y) \to F(x)) \to \forall x{\prec}\overline{m}\ F(x).$$

   Point out which theorems from the book you have used.

8. Write down a pseudo-Π^1_1-sentence whose truth-complexity is ε_0 and whence, which is not provable in PA. Point out which theorems from the book you have used.

2 Interpolation

We enrich the Tait language for predicate logic by two symbols $\top$ and $\bot$ where the intended meaning is that $\top$ is always true and $\bot$ is always false. Consequently we define $\sim \bot := \top$ and $\sim \top := \bot$.

1. Write down a sound new axiom for these constants. (Hint: you only need to take care of $\top$.)

2. We define a new calculus for predicate logic in the language with $\bot$ and $\top$ to be exactly the regular Tait-calculus for predicate logic enriched by the axiom you have defined here above. Let us denote derivability in the new calculus by $\Vdash^n_T$. Prove that $\Vdash^n_T \Delta, \bot$ implies $\Vdash^n_T \Delta$. (Hint: if you have problems proving this, you might have the wrong rule in which case you can contact me so that you can make the rest of this exercise.)

3. Prove that the newly defined Tait calculus is sound.

4. Prove that this new Tait calculus is also complete in the enriched language. (Hint: we know it is complete for sets that do not contain $\bot$ or $\top$.)

5. Prove that if $\Vdash^n_T \Gamma, \Delta$ then there is a formula F such that F only contains constant, function and relation symbols that occur both in Γ and in Δ (possibly with the sole exception of $\bot$ and $\top$) and so that moreover we have both $\Vdash^k_T \Gamma, F$ and $\Vdash^l_T \Delta, \sim F$ for some natural numbers k and l. Such a formula F is called an *interpolant*.

3 Natural Deduction

Let us work in predicate logic where we only allow the connectives $\wedge$, $\to$ and $\forall$. As usual, $\neg A$ is defined as $A \to \bot$. We consider a fragment of natural deduction where we only have the rules for $\to E$, $\to I$, $\wedge I$, $\wedge E$, $\forall I$, $\forall E$, $\bot$ (ex falso) and RAA.

1. We can define $A \vee B$ as usual as $\neg(\neg A \wedge \neg B)$. Show that we can derive the $\vee I$ rule in our restricted system. That is, if we have a proof

$$\frac{\Pi}{A}$$

   then we can obtain a proof Π' of $\neg(\neg A \wedge \neg B)$ so that $\mathcal{A}(\Pi')$ –the open assumptions in Π'– equals $\mathcal{A}(\Pi)$.

2. As mentioned, we can define $A \vee B$ as usual as $\neg(\neg A \wedge \neg B)$. Show that we can derive the $\vee E$ rule in our restricted system. That is, if we have a proof

$$\frac{\Pi}{\neg(\neg A \wedge \neg B)}$$

and proofs

$$\frac{\Pi_1}{\mathrm{C}}$$

and

$$\frac{\Pi_2}{\mathrm{C}}$$

then we can obtain a proof Π' of C so that $\mathcal{A}(\Pi')$ –the open assumptions in Π'– equals $\mathcal{A}(\Pi) \cup (\mathcal{A}(\Pi_1) \setminus \{A\}) \cup (\mathcal{A}(\Pi_2) \setminus \{B\})$.

3. Reason to the effect that the previous two exercises show that we can completely define $\vee$ in our restricted fragment of natural deduction.

4. Prove that we can completely define $\exists$ in our restricted calculus.

5. Prove that in our restricted calculus we can move applications of the ex-falso rule "upwards". That is to say, we may assume that applications of the ex-falso rule are restricted to atomic formulas:

$$\frac{\begin{array}{c}\mathcal{D}\\ \bot\end{array}}{A}\ \bot$$

with A atomic.

6. Prove that in our restricted calculus, we can assume wlog that each RAA rule is only applied to atomic assumptions. That is

$$\frac{\begin{array}{c}\dfrac{[\neg A]^1}{\mathcal{D}}\\ \bot\end{array}}{A}\ \mathsf{RAA}, 1$$

with A atomic.

4 Cut elimination Tait calculus

Make Exercise 4.5.3 from the book.

5 Hydra battle

Make Exercise 4.4.14 from the book.

6 Simultaneous inductive definitions

Make Exercise 6.4.10 from the book.