Proof Theory an Automated Theorem Proving 2013
Exercises
Week 4

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1 Structural rule

We have already proven the admissibility of the structural rule: If $\vdash^m_{\mathsf{T}} \Gamma$ then $\vdash^m_{\mathsf{T}} \Gamma, \Delta$ (we were actually allowed to increase the m too).

1. Sometimes the structure of a proof witnessing $\vdash^m_{\mathsf{T}} \Gamma, \Delta$ is different from the structure of $\vdash^m_{\mathsf{T}} \Gamma$. Illustrate this using the example $\vdash_{\mathsf{T}} A \vee \sim A, A$.

2. Under certain conditions though, we do have that the structure of the proof remains the same after weakening. Formulate and prove a statement to the effect that from $\vdash^m_{\mathsf{T}} \Gamma$ we can conclude $\vdash^m_{\mathsf{T}} \Gamma, \Delta$ by a proof that shares the same structure as the original proof of $\vdash^m_{\mathsf{T}} \Gamma$. What is a sufficient condition on Δ?

2 Traces

Throughout all this exercise suppose that we split a provable sequent in two disjunct parts Γ and Δ; Thus, $\vdash^m_{\mathsf{T}} \Gamma, \Delta$ for some $m \in \omega$.

A proof Π of Γ, Δ is a well-founded labeled binary tree such that $\langle \ \rangle \in \Pi$ with label $\delta(\Delta)$ and so that each s is a finite sequence of 0 and 1's so that if $s \in \Pi$ then,

1. if $s^\frown\langle n \rangle$ implies $n = 0$, then $\frac{\delta(s^\frown\langle 0 \rangle)}{\delta(s)}$ is an application of one of the rules $(\exists), (\forall), \vee$,

2. if s is a leaf, then $\delta(s)$ is an axiom,

3. if both $s^\frown\langle 0 \rangle$ and $s^\frown\langle 1 \rangle$ are in Π, then $\frac{\delta(s^\frown\langle 0 \rangle) \qquad \delta(s^\frown\langle 1 \rangle)}{\delta(s)}$ is an application of the $(\wedge)$ rule.

Given a proof Π, we wish to give a sensible definition of $\mathsf{Trace}(\Gamma)$. Basically, $\mathsf{Trace}(\Gamma)$ should tell us which formulas "in" Π originate in Γ. We define $\mathsf{Trace}(\Gamma) = \Pi$ but now with a different labeling function τ. We inductively define $\tau(\langle\ \ \rangle) = \Gamma$, and if $s \in \Pi$, then

1. if $s^\frown\langle n\rangle$ implies $n = 0$, and $\frac{\delta(s^\frown\langle 0\rangle)}{\delta(s)}$ is an application of the rule $(\vee)$, with $\tau(s) = A_0 \vee A_1, \Gamma'$ where $A_0 \vee A_1$ is the newly introduced formula, and A_i is the premiss for this newly introduced formula then $\tau(s^\frown\langle 0\rangle) := \Gamma' \cup (\{A_i, A_0 \vee A_1\} \cap \delta(s^\frown\langle 0\rangle))$
2. etc.

Actual exercise:

1. Write down the other cases of the definition of $\mathsf{Trace}(\Gamma)$.
2. We call an axiom $\Delta', A, \sim A$ *mixed* in case $A \in \mathsf{Trace}(\Gamma)$ and $\sim A \in \mathsf{Trace}(\Delta)$ for some atomic formula A. Show that if Π does not contain mixed axioms, but only axioms of the form $\Gamma', A, \sim A$ with $\{\sim A, A\} \subseteq \mathsf{Trace}(\Gamma)$, then $\vdash^m_{\mathsf{T}} \Gamma$.