Proof Theory an Automated Theorem Proving 2013 Exercises Week 7b

Lecturer: Joost J. Joosten

April 5, 2013

1 Accessible parts

Let $\prec$ be a binary relation on the naturals. Prove that $\prec$ restricted to $\mathsf{Acc}_{\prec}$ defines a well-founded order.

2 Essentially transfinite proofs

1. Let $\mathcal{L}$ be the language of arithmetic with function symbols $\{+, \cdot, \mod 2, S\}$, constant 0 and relation symbols $\{=, \leq, <\}$ where the familiar symbols are interpreted as always and $\mod 2$ is a unary function mapping a number to its rest after division by two (so $x \mod 2 = 1$ in case x is odd and 0 otherwise). We can use this language to easily say that odd numbers are "bigger" than even numbers, and that odd numbers between them are compared according to the regular less-than relation and likewise for even numbers.

   Write down an arithmetic formula $\varphi(x, y)$ in the language $\mathcal{L}$ that defines (along the lines sketched above) a primitive recursive well-order $\prec$ on $\mathbb{N}$ so that moreover $\mathsf{field}(\prec) = \mathbb{N}$ and $\mathsf{otyp}(\prec) = \omega \cdot 2$.

2. Recall that in our enriched language, we have a symbol for the (characteristic function of the) above defined relation $\prec$. In the remainder of this exercise, we will denote this relation by $\prec$. For any ordinal $\alpha < \omega \cdot 2$, let $\tilde{\alpha}$ denote the natural number n so that $\mathsf{otyp}_{\prec}(n) = \alpha$.

   (a) Compute $\widetilde{\omega + 5}$.

   (b) Since $\mathsf{field}(\prec) = \mathbb{N}$, we let $\mathsf{prog}_{\prec}(X) := \forall x\ (\forall y{\prec}x\ y \in X \to x \in X)$. Write $\neg\mathsf{prog}_{\prec}(X)$ in the Tait language.

(c) Recall that $\forall\, y \prec x \; y \in X$ is short for $\forall y \; (y \prec x \rightarrow y \in X)$. In our infinitary verification calculus, give a proof of

$$\neg \mathsf{prog}_{\prec}(X), \tilde{0} \in X$$

and determine the height of this proof.

(d) For $n \in \omega$, inductively describe a proof of

$$\neg \mathsf{prog}_{\prec}(X), \tilde{n} \in X$$

in our infinitary verification calculus and estimate the height of such a proof as a function of n.

(e) Describe a proof of

$$\neg \mathsf{prog}_{\prec}(X), \tilde{\omega} \in X$$

in our infinitary verification calculus.

(f) What is the truth-complexity of $\mathsf{Prog}_{\prec}(X) \rightarrow \forall x \; x \in X$? Motivate your answer.