Exercises Set 1 Proof Theory, 2019

Exercise 0.1. *1. What rule(s) should be added to* G3cp *so that we can add the logical constant* $\top$ *to our language and obtain a system* G3cp$\top$.

2. Prove soundness and completeness of your proposed system G3cp$\top$.

Exercise 0.2. *There are two obvious ways to work with sequents* $\Gamma \Longrightarrow \Delta$. *One way is to say that both* Γ *and* Δ *are multi-sets and the other is to say that both should be sets. However, there is a middle way. Here, we require that* Γ *and* Δ *are sets but allow for repetition in the rule formulation. For example, in the rule*

$$R{\rightarrow}\ \frac{\Gamma, A \Longrightarrow \Delta, B}{\Gamma \Longrightarrow \Delta, A \rightarrow B}\ .$$

we allow for $A \in \Gamma$ *or* $A \notin \Gamma$ *where the first allows for repeating a single formula. Only in this system, the following would be a proof:*

$$\dfrac{\dfrac{\dfrac{\dfrac{P \Longrightarrow \bot, P \qquad \bot, P \Longrightarrow \bot}{P, \neg P \Longrightarrow \bot, \bot}\,L{\rightarrow}}{P \Longrightarrow \neg\neg P, \bot}\,R{\rightarrow}}{\Longrightarrow \neg\neg P, \neg P}\,R{\rightarrow}}{\Longrightarrow \neg\neg P \vee \neg P}\,R{\vee}\ .$$

Show that for G3c *all three readings are sound and complete.*

Exercise 0.3. *We can define a constructive calculus where conclusions can have succedents with more than one element. However, we should then replace the rule for implication introduction on the right to*

$$R{\rightarrow}\ \frac{\Gamma, A \Longrightarrow \Delta}{\Gamma \Longrightarrow A \rightarrow \bigvee \Delta}\ .$$

Do we have soundness and completeness for the resulting proof system(s)?

Exercise 0.4. *Prove the following tautology in* G3c*:*

$$\forall x \neg R(x,x) \wedge \forall x \forall y \forall z\ (R(x,y) \wedge R(y,z) \rightarrow R(x,z)) \rightarrow \neg \exists x \exists y (R(x,y) \wedge R(y,x)).$$

Exercise 0.5. *In this exercise you are asked to retrieve countermodels from non-provable sequents for classical predicate calculus. As such you are asked to apply rules to our non-provable sequents so that the resulting trees which are actually not proofs suggest a counter model. Do this for the following formulas:*

1. $\exists x Ax \rightarrow \forall x Ax$;
2. $\forall x \neg R(x,x) \wedge \forall x \forall y \forall z\ (R(x,y) \wedge R(y,z) \rightarrow R(x,z)) \rightarrow \exists x \exists y (R(x,y) \wedge R(y,x))$;
3. $\exists x \exists y R(x,y) \wedge \forall x \neg R(x,x) \wedge \forall x \forall y \forall z\ (R(x,y) \wedge R(y,z) \rightarrow R(x,z)) \rightarrow \exists x \exists y (R(x,y) \wedge R(y,x))$;

Observe that the second formula has a finite countermodel. Can this be read-off from a non-terminating proof?

Exercise 0.6. *In this exercise we consider classical first-order logic corresponding to the system* G3c*. In case a formula is non-provable, we can still apply our rules to obtain a tree that –apart from possible leafs– locally satisfies the rules. Possibly such a tree will have infinite branches. Show how trees corresponding to non-provable formulas give rise to a model falsifying this formula and that this thus constitutes for a completeness proof. Hint: show that a model with an assignment falsifying an antecedent of a rule will also falsify the conclusion of a rule. Falsifying a sequent should be properly defined. In case all the leaves are axioms, one will have to resort to an infinite branch to finish the proof. In a sense, the union of such an infinite branch can be used to define a term model. For the sake of a smooth argument it may be good to work with a countable set of variables used for the quantifiers, a disjoint countable set of variables used for the free variables, and a countable set of constants used to define the term model.*

Exercise 0.7. *Consider the previous exercise. It is clear that different infinite branches may lead to different (non-isomorphic) models. Can all possible countable models of a non-provable formula be obtained by the above procedure?*