Lògica 2016–2017, (Code 360961)

Practice second partial exam

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Exercise 0

Give derivations of the following tautologies

1. $\Big(\big((A \to B) \to B\big) \to B\Big) \to (A \to B)$;
2. $\neg(A \wedge B) \to \neg A \vee \neg B$;
3. $\neg A \vee \neg B \to \neg(A \wedge B)$;
4. $\neg(A \vee B) \to \neg A \wedge \neg B$;
5. $\neg A \wedge \neg B \to \neg(A \vee B)$;
6. $A \vee B \to B \vee A$.

Exercise 1

For each of the following statements, say if they are true or not. In case $\varphi \models \psi$ is a logical consequence, give a derivation of the formula

$$\varphi \to \psi$$

and if we do not have logical consequence exhibit a counter-model.

1. $\forall x\, P(x) \to Qc \models \forall x\ (P(x) \to Qc)$;
2. $\forall x\ (P(x) \to Qc) \models \forall x\, P(x) \to Qc$;
3. $\exists x\, P(x) \to Qc \models \exists x\ (P(x) \to Qc)$;
4. $\exists x\ (P(x) \to Qc) \models \exists x\, P(x) \to Qc$;

Exercise 1.5

Write each of the following formulas in Prenex Normal form:

1. $(\forall x\, P(x) \to Qc) \to (\forall x\ (P(x) \to Qc))$;
2. $(\forall x\ (P(x) \to Qc)) \to (\forall x\, P(x) \to Qc)$;
3. $(\exists x\, P(x) \to Qc) \to (\exists x\ (P(x) \to Qc))$;
4. $(\exists x\ (P(x) \to Qc)) \to (\exists x\, P(x) \to Qc)$;

Exercise 2

Using the language with only a binary relation R, give a finite collection Γ of sentences so that any model $\mathcal{A}$ that satisfies all sentences in Γ has to be infinite.

Exercise 3

1. State the soundness theorem of our derivation calculus for first order logic.
2. State the completeness theorem of our derivation calculus for first order logic.

Exercise 4

Let $\mathcal{M}$ be a model and R and S be a relations on the domain of $\mathcal{M}$. Show that

1. if R is definable on $\mathcal{M}$, then so is $\mathsf{dom}(R)$ (the domain of R);
2. if R is definable on $\mathcal{M}$, then so is $\mathsf{ran}(R)$ (the range of R);
3. if R is definable on $\mathcal{M}$, then so is $\mathsf{field}(R)$ (the field of R);
4. if R is definable on $\mathcal{M}$, then so is R^{-1} (the converse relation of R);
5. if R and S are definable on $\mathcal{M}$, then so is $R;S$ (the compositional product of R and S);

Exercise 5

We consider the language $\mathcal{L} := \{\overline{0}, S, \prec\}$ where $\overline{0}$ is a constant, and both S and $\prec$ are binary relations. Moreover, we consider two models $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{L}$ defined by:

$$\mathcal{A} := \langle \mathbb{N}, 0, \{\langle a, b\rangle \mid b = a + 1\}, \{\langle a, b\rangle \mid a < b\}\rangle$$

and

$$\mathcal{B} := \langle \mathbb{N} \cup \{\omega\}, 0, \{\langle a, b\rangle \mid b = a + 1\}, \{\langle a, b\rangle \mid a < b\} \cup \{\langle a, \omega\rangle \mid a \in \mathbb{N}\}.\rangle$$

Here, $\mathbb{N}$ stands for the set of natural numbers $\{0, 1, 2, 3, 4, \ldots\}$ and $<$ and $+$ denote the regular less-than ordering and addition respectively on $\mathbb{N}$. Next, we consider $\mathcal{L}' := \mathcal{L} \cup \{P\}$ where P is a predicate. We consider the formulas

$$\varphi := \Big(P(\overline{0}) \wedge \forall x, y \ (\big(P(x) \wedge S(x, y)\big) \to P(y))\Big) \to \forall z \ P(z);$$

and

$$\psi := \forall x \ (\forall y, x \ \big([\prec (y, x) \to P(y)] \to P(x)\big)) \to \forall z \ P(z).$$

1. Show that for any model $\mathcal{C} := \langle X, \overline{0}^{\mathcal{C}}, S^{\mathcal{C}}, \prec^{\mathcal{C}}, P^{\mathcal{C}}\rangle$ of $\mathcal{L}'$ we have that if $\langle X, \overline{0}^{\mathcal{C}}, S^{\mathcal{C}}, \prec^{\mathcal{C}}\rangle = \mathcal{A}$, then $\mathcal{C} \models \varphi$.
2. Show that for any model $\mathcal{C} := \langle X, \overline{0}^{\mathcal{C}}, S^{\mathcal{C}}, \prec^{\mathcal{C}}, P^{\mathcal{C}}\rangle$ of $\mathcal{L}'$ we have that if $\langle X, \overline{0}^{\mathcal{C}}, S^{\mathcal{C}}, \prec^{\mathcal{C}}\rangle = \mathcal{A}$, then $\mathcal{C} \models \psi$.
3. Show that for any model $\mathcal{C} := \langle X, \overline{0}^{\mathcal{C}}, S^{\mathcal{C}}, \prec^{\mathcal{C}}, P^{\mathcal{C}}\rangle$ of $\mathcal{L}'$ we have that if $\langle X, \overline{0}^{\mathcal{C}}, S^{\mathcal{C}}, \prec^{\mathcal{C}}\rangle = \mathcal{B}$, then $\mathcal{C} \models \psi$.
4. Show that there is some model $\mathcal{C} := \langle X, \overline{0}^{\mathcal{C}}, S^{\mathcal{C}}, \prec^{\mathcal{C}}, P^{\mathcal{C}}\rangle$ of $\mathcal{L}'$ so that $\langle X, \overline{0}^{\mathcal{C}}, S^{\mathcal{C}}, \prec^{\mathcal{C}}\rangle = \mathcal{B}$, and $\mathcal{C} \models \neg\varphi \wedge \psi$.