Lògica 2015–2016, (Code 360961)

Practice second partial exam

Lecturer: Joost J. Joosten

Friday, November 11, 2015

Exercise 0

Give derivations of the following sequents

1. $\Big(\big((A \to B) \to B\big) \to B\Big) \to (A \to B)$;
2. $\neg(A \wedge B) \qquad \neg A \vee \neg B$;
3. $\neg A \vee \neg B \qquad \neg(A \wedge B)$;
4. $\neg(A \vee B) \qquad \neg A \wedge \neg B$;
5. $\neg A \wedge \neg B \qquad \neg(A \vee B)$;
6. $A \vee B \qquad B \vee A$.

Exercise 1

For each of the following statements, say if they are true or not. In case $\varphi \models \psi$ is a logical consequence, give a derivation of the sequent

$$\varphi \qquad\qquad \psi$$

and if we do not have logical consequence exhibit a counter-model.

1. $\forall x\, P(x) \to Qc \models \forall x\ (P(x) \to Qc)$;
2. $\forall x\ (P(x) \to Qc) \models \forall x\, P(x) \to Qc$;
3. $\exists x\, P(x) \to Qc \models \exists x\ (P(x) \to Qc)$;
4. $\exists x\ (P(x) \to Qc) \models \exists x\, P(x) \to Qc$;

Exercise 2

Using the language with only a binary relation R, give a finite collection Γ of sentences so that any model $\mathcal{A}$ that satisfies all sentences in Γ has to be infinite.

Exercise 3

1. State the soundness theorem of our derivation calculus for first order logic.
2. State the completeness theorem of our derivation calculus for first order logic.

Exercise 4

We consider the language $\mathcal{L} := \{\overline{0}, S, \prec\}$ where $\overline{0}$ is a constant, and both S and $\prec$ are binary relations. Moreover, we consider two models $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{L}$ defined by:

$$\mathcal{A} := \langle \mathbb{N}, 0, \{\langle a, b\rangle \mid b = a + 1\}, \{\langle a, b\rangle \mid a < b\}\rangle$$

and

$$\mathcal{B} := \langle \mathbb{N} \cup \{\omega\}, 0, \{\langle a, b\rangle \mid b = a + 1\}, \{\langle a, b\rangle \mid a < b\} \cup \{\langle a, \omega\rangle \mid a \in \mathbb{N}\}.\rangle$$

Here, $\mathbb{N}$ stands for the set of natural numbers $\{0, 1, 2, 3, 4, \ldots\}$ and $<$ and $+$ denote the regular less-than ordering and addition respectively on $\mathbb{N}$. Next, we consider $\mathcal{L}' := \mathcal{L} \cup \{P\}$ where P is a predicate. We consider the formulas

$$\varphi := \Big(P(\overline{0}) \wedge \forall x, y\ \big(\big(P(x) \wedge S(x, y)\big) \to P(y)\big)\Big) \to \forall z\ P(z);$$

and

$$\psi := \forall x\ (\forall y, x\ \big([\prec (y, x) \to P(y)] \to P(x)\big)) \to \forall z\ P(z).$$

1. Show that for any model $\mathcal{C} := \langle X, \overline{0}^{\mathcal{C}}, S^{\mathcal{C}}, \prec^{\mathcal{C}}, P^{\mathcal{C}}\rangle$ of $\mathcal{L}'$ we have that if $\langle X, \overline{0}^{\mathcal{C}}, S^{\mathcal{C}}, \prec^{\mathcal{C}}\rangle = \mathcal{A}$, then $\mathcal{C} \models \varphi$.
2. Show that for any model $\mathcal{C} := \langle X, \overline{0}^{\mathcal{C}}, S^{\mathcal{C}}, \prec^{\mathcal{C}}, P^{\mathcal{C}}\rangle$ of $\mathcal{L}'$ we have that if $\langle X, \overline{0}^{\mathcal{C}}, S^{\mathcal{C}}, \prec^{\mathcal{C}}\rangle = \mathcal{A}$, then $\mathcal{C} \models \psi$.
3. Show that for any model $\mathcal{C} := \langle X, \overline{0}^{\mathcal{C}}, S^{\mathcal{C}}, \prec^{\mathcal{C}}, P^{\mathcal{C}}\rangle$ of $\mathcal{L}'$ we have that if $\langle X, \overline{0}^{\mathcal{C}}, S^{\mathcal{C}}, \prec^{\mathcal{C}}\rangle = \mathcal{B}$, then $\mathcal{C} \models \psi$.
4. Show that there is some model $\mathcal{C} := \langle X, \overline{0}^{\mathcal{C}}, S^{\mathcal{C}}, \prec^{\mathcal{C}}, P^{\mathcal{C}}\rangle$ of $\mathcal{L}'$ so that $\langle X, \overline{0}^{\mathcal{C}}, S^{\mathcal{C}}, \prec^{\mathcal{C}}\rangle = \mathcal{B}$, and $\mathcal{C} \models \neg\varphi \wedge \psi$.