Arithmetics

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Suggestions for exercises.

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Here are some suggestions for exercises. Consult your tutor on which exercises you could do. Filling in details in the notes also can serve as an exercise.

1. Prove that every formula φ in the predicate logic is equivalent to one in prenex normal form.

2. Show that $B\Sigma_n$ is finitely axiomatizable.

3. Exercise 7.3 of Kaye. Part b should read $\mathrm{L}\Pi_n \vdash \mathrm{I}\Sigma_n$ instead of $\mathrm{L}\Pi_n \vdash \mathrm{L}\Sigma_n$ (which is by the way also the case). Make use of proposition 7.2. At the d. part for proving $\mathrm{I}\Sigma_n \vdash \mathrm{I}\Pi_n$ consider $u \le a \to \neg\varphi(a-u)$.

4. Give two Σ_1 numerations of a theory such that the corresponding proof predicates are not provably equivalent over say EA. (Hint: consider an enumeration of the axioms until you find a proof of inconsistency.)

5. What is the logical complexity of the Rfn_{Π_n} axioms? And for the other restricted reflection axioms?

6. Exercise 5.11 from Kaye showing that the Ackermann function is not primitive recursive.

7. Show that $\mathrm{I}\Sigma_1 + \mathrm{I}\Pi_2^-$ proves the totality of the Ackermann function.

8. In what sense can we say that $\mathrm{PRA} \equiv \mathrm{EA} + \Sigma_1\text{-IR}$.

9. Prove Herbrand's theorem. There is a model theoretic proof and a prooftheoretic one.

10. Show that $\Sigma_1\text{-IR} \equiv \Pi_2\text{-IR}$.

11. Show that $\mathrm{EA} + \Sigma_n\text{-IR}^- \vdash \mathrm{B}\Sigma_n - \mathrm{R}^-$.

12. Consider the proof in Boolos of provable Σ_1 completeness on pages 46-49. Indicate where changes should be adapted if we want to have provable Σ_2 completeness for [1].

13. Show that GL is sound for the $[n]$ interpretation.

14. If one tries to adapt the proof of Σ_n-IR $\equiv$ Σ_n-IR$^-$ to make it suit for $\mathrm{I}\Sigma_n^- \vdash \mathrm{I}\Sigma_n$, at what point would this break down?

15. Show that $\mathrm{I}\Sigma_n' \equiv \mathrm{I}\Sigma_n$ where $\mathrm{I}\Sigma_n'$ is defined by adding axioms of the form $\forall x[\varphi(0) \wedge \forall y < x(\varphi(y) \to \varphi(y+1)) \to \varphi(x)]$. $(varphi \in \Sigma_n.)$ What about the parameter free case?

16. Show that if $I \subset_e M$ and $M \models \pi$ with $\pi \in \Pi_1$ then $I \models \pi$.

17. Do we have $f(x) = O(g(x)) \to 2^{f(|x|)} = O(2^{g(|x|)})$?

18. Prove that $\omega_n(x)$ tends to 2^x as n goes to infinity.

19. Assuming that 2^n is Δ_0 definable show that each $\omega_n(x)$ is Δ_0 definable. Each term using only plus times and ω_n is eventually majorized by ω_{n+1}. Generalize Parikh's theorem to show that $\mathrm{I}\Delta_0 + \Omega_n \nvdash \Omega_{n+1}$.

20. Show that proper cuts of $\mathrm{I}\Sigma_{n+1}$ exist. If possible provide one.

21. Proof fact 5.6.

22. Fill in the details of Solovay's theorem on the shortening of cuts.