A decidable fragment of the Quantified Provability Logic of Heyting Arithmetic

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Formalised provability and applications

- Provability is a central notion in logic and metamathematics
- For theories like PA we can write a Σ_1 predicate $\Box_{\mathsf{PA}}(\cdot)$ such that

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  $\mathsf{PA} \vdash \lambda \leftrightarrow \neg \Box_{\mathsf{PA}}(\ulcorner 0 = 1 \urcorner)$, that is
  $\mathsf{PA} \vdash \lambda \leftrightarrow \big(\Box_{\mathsf{PA}}(\ulcorner 0 = 1 \urcorner) \to 0 = 1\big)$

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Theorem

The $\Box_{\mathsf{PA}}(\cdot)$ predicate is Σ^0_1-complete. That is, for each c.e. set A, there is an arithmetical formula $\rho_A(x)$ such that

$$A = \{n \in \mathbb{N} \mid \mathbb{N} \models \Box_{\mathsf{PA}}\Big(\rho_A(n)\Big)\}.$$

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- Löb's Theorem:

  If $\mathsf{PA} \vdash \Box_{\mathsf{PA}}(\ulcorner A \urcorner) \to A$, *then* $\mathsf{PA} \vdash A$, *for any* PA *formula* A

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- Formalised Löb's Theorem (ignoring GNs):

$$\mathsf{PA} \vdash \Box_{\mathsf{PA}}\Big(\Box_{\mathsf{PA}} A \to A\Big) \to \Box_{\mathsf{PA}} A$$

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  for any PA *formula* A
- Characterise all provably structural properties in two steps
  - $\mathcal{L}_{\Box}$ with $\mathrm{Form}_{\Box} := \bot \mid \mathrm{Prop} \mid \mathrm{Form}_{\Box} \to \mathrm{Form}_{\Box} \mid \Box \mathrm{Form}_{\Box}$
  - Define a denotation of $\mathcal{L}_{\Box}$ formulas inside the $\mathcal{L}_{\mathsf{PA}}$ formulas

Arithmetical realizations

An arithmetical realization is any function $(\cdot)^\star$ taking:

$$\text{formulas in } \mathcal{L}_\Box \to \text{sentences in } \mathcal{L}_{\mathsf{PA}}$$
$$\text{propositional variables} \to \text{arithmetical sentences}$$
$$\text{boolean connectives} \to \text{boolean connectives}$$
$$\Box \to \Box_{\mathsf{PA}}$$

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Clearly, for any realization $(\cdot)^\star$ we have for example

$$\mathsf{PA} \vdash \Big(\Box(p \to q) \to (\Box p \to \Box q)\Big)^\star$$

since

$$\mathsf{PA} \vdash \Box_{\mathsf{PA}}(p^\star \to q^\star) \to (\Box_{\mathsf{PA}} p^\star \to \Box_{\mathsf{PA}} q^\star)$$

regardless of $(\cdot)^\star$

The Provability Logic of a Theory

- For a c.e. theory T we define

$$\mathrm{PL}(T) := \{\varphi \in \mathcal{L}_\Box \mid \text{for any } (\cdot)^\star, \text{ we have } T \vdash (\varphi)^\star\}$$

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A candidate

- GL is the normal modal logic with axioms
  - All classical logical tautologies in $\mathcal{L}_\Box$ like $\Box p \vee \neg\Box p$, etc.
  - All distributions axioms: $\Box(A \to B) \to (\Box A \to \Box B)$,
  - All Löb axioms: $\Box(\Box A \to A) \to \Box A$.
- and rules
  - Modus Ponens $\dfrac{A \to B \quad A}{B}$,
  - Necessitation $\dfrac{A}{\Box A}$.

Solovay's Theorem

Theorem (Solovay, 1976)

Let $\varphi \in \mathcal{L}_\Box$. *Then:*

$$\mathsf{GL} \vdash \varphi$$
$$\Updownarrow$$
$$\mathsf{PA} \vdash (\varphi)^\star \text{ for any arithmetical realization } (\cdot)^\star$$

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Thus, even though PL(PA) is *prima facie* of complexity Π^0_2, it allows for a decidable description

$$\mathsf{GL} = \{\varphi \in \mathcal{L}_\Box \mid \text{for any } (\cdot)^\star, \text{ we have } \mathsf{PA} \vdash (\varphi)^\star\}$$

of complexity PSPACE.

True provability logic

- $\mathsf{PA} \nvdash \Box_{\mathsf{PA}}(\ulcorner 0 = 1 \urcorner) \to 0 = 1$
- $\mathbb{N} \models \Box_{\mathsf{PA}}(\ulcorner \varphi \urcorner) \to \varphi$ for whatever sentence φ

For a c.e. theory T we define

$$\mathsf{TPL}(T) := \{\varphi \in \mathcal{L}_{\Box} \mid \text{for any } (\cdot)^{\star}, \text{ we have } \mathbb{N} \models (\varphi)^{\star}\}$$

A priori, complexity above true arithmetic.
However,

$$\mathsf{TPL}(\mathsf{PA}) = \mathsf{GLS}.$$

Here GLS is axiomatised by all theorems of GL and all reflection axioms $\Box A \to A$ with MP as the only rule.

Solovay for quantified modal logic?

Let $\mathcal{L}_{\Box,\forall}$ be the language of relational quantified modal logic:

$$\top, \text{ relation symbols, boolean connectives, } \forall x, \text{ and } \Box$$

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Define arithmetical realizations $(\cdot)^\bullet$ for $\mathcal{L}_{\Box,\forall}$:

$$\text{formulas in } \mathcal{L}_{\Box,\forall} \to \text{formulas in } \mathcal{L}_{\mathrm{PA}}$$

$$n\text{-ary relation symbols} \to \text{arithmetical formulas with } n \text{ free variables}$$

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For a c.e. theory T we define

$$\mathsf{QPL}(T) := \{\varphi \in \mathcal{L}_{\Box,\forall} \mid \text{for any } (\cdot)^\bullet, \text{ we have } T \vdash (\varphi)^\bullet\}$$

and

$$\mathsf{TQPL}(T) := \{\varphi \in \mathcal{L}_{\Box,\forall} \mid \text{for any } (\cdot)^\bullet, \text{ we have } \mathbb{N} \models (\varphi)^\bullet\}$$

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Example: $\Box\forall x P(x) \to \forall x \Box P(\dot{x})$

A Degenerate Quantified Provability Logic

If we define $\mathsf{QL}(T) = \{\varphi \in \mathcal{L}_{\mathsf{fol}} \mid \text{for any } (\cdot)^{\bullet}, \text{ we have } T \vdash (\varphi)^{\bullet}\}$, then it is not hard to see that $\mathsf{CQC} = \mathsf{QL}(\mathsf{PA})$.
Proof:

- $\subseteq$ if $\pi \vdash_{\mathsf{CQC}} \varphi$, then also $\pi^{\bullet} \vdash_{\mathsf{CQC}} \varphi^{\bullet}$, whence $\pi^{\bullet} \vdash_{\mathsf{PA}} \varphi^{\bullet}$
- $\supseteq$ Henkin construction in arithmetic

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$$\mathrm{QPL}(\mathrm{PA} + \mathrm{Incon}(\mathrm{PA})) = \mathrm{CQC} + \Box\bot$$

Negative results

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Theorem (Vardanyan, 1986 and McGee, 1985)

$$\{\textit{closed } \varphi \in \mathcal{L}_{\Box,\forall} \mid \textit{for any } (\cdot)^{\bullet}, \ \textit{we have } \mathsf{PA} \vdash (\varphi)^{\bullet}\}$$

is Π^0_2*-complete. Thus it is not recursively axiomatisable.*

Theorem (Artemov, 1985)

TQPL(PA) *is not arithmetical.*

Theorem (Vardanyan, 1985)

TQPL(PA) *is* Π^0_1 *complete in true arithmetic.*

Artemov's Lemma

- Let $F \in \mathcal{L}_{\mathsf{PA}}$ be a formula

$\mathcal{L}_{PA}$

F

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- Let $F \in \mathcal{L}_{\mathsf{PA}}$ be a formula
- Replace arithmetical symbols $0, +1, +, \times, =$ with predicates Z, S, A, M, E, obtaining $\{F\} \in \mathcal{L}_{\forall}$

$$\begin{tikzcd}
\mathcal{L}_{PA} & \mathcal{L}_{\forall} \\
F \arrow[dr, bend left] & \\
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When are F and $\{F\}^\bullet$ equivalent over PA?

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F \arrow[dr, bend left] \arrow[dd, phantom, "\vdash\updownarrow"] & \\
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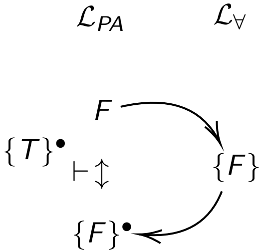

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\begin{matrix}\{T\}^\bullet \\ D^\bullet\end{matrix} \ \vdash \updownarrow & & \{F\} \arrow[ld, bend left] \\
\{F\}^\bullet & &
\end{tikzcd}$$

$$\begin{aligned}
D := {} & \Diamond\top \wedge \\
& \forall x\,(Z(x) \to \Box Z(x)) \wedge \forall x\,(\neg Z(x) \to \Box\neg Z(x)) \wedge \\
& \cdots S \cdots A \cdots M \cdots E
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When are F and $\{F\}^\bullet$ equivalent over PA?

- Under $\{T\}^\bullet$ to get arithmetical axioms...
- ... and under $D^\bullet$ to get recursive $A^\bullet$ and $M^\bullet$
- By Tennenbaum's Theorem the model induced by $(\cdot)^\bullet$ is standard, hence $\mathbb{N} \models S \iff (\{T\} \wedge D \to \{S\}) \in \mathrm{TQPL(PA)}$

$$
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Berarducci ('89) : $\{\varphi \in \mathcal{L}_{\Box,\forall} \mid$ for any $(\cdot)^\bullet \in \Sigma^0_1$, we have $\mathsf{PA} \vdash (\varphi)^\bullet\}$ is Π^0_2-complete.

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One easily sees that $\mathsf{QPL}(\mathsf{PA} + \Box_{\mathsf{PA}}\bot)$ is r.e., but it seems that $\mathsf{QPL}(\mathsf{PA} + \Box_{\mathsf{PA}}\Box_{\mathsf{PA}}\bot)$ is also Π^0_2-complete.

Theorem (Visser, de Jonge, 2006)

$\mathsf{QPL}(T)$ *is* Π^0_2 *complete for any* Σ_1*-sound theory* T *extending* EA*.*

Archive for Mathematical Logic 2006: No Escape from Vardanyan's

Restricted signatures and logics: RC_1

Restrict $\mathcal{L}_\Box$ to the strictly positive fragment $\mathcal{L}_\Diamond$:

$$\mathcal{L}_\Diamond ::= \top \mid \varphi \wedge \varphi \mid \Diamond\varphi$$

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Define a calculus RC_1 with statements $\varphi \vdash_{\mathsf{RC}_1} \psi$ where:

$$\varphi, \psi \in \mathcal{L}_\Diamond$$

RC_1: Axioms and rules

$$\varphi \vdash \top \qquad \varphi \wedge \psi \vdash \varphi$$

$$\varphi \vdash \varphi \qquad \varphi \wedge \psi \vdash \psi$$

$$\frac{\varphi \vdash \psi \quad \psi \vdash \chi}{\varphi \vdash \chi} \qquad \frac{\varphi \vdash \psi \quad \varphi \vdash \chi}{\varphi \vdash \psi \wedge \chi}$$

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RC_1 Main result

Theorem (Dashkov, Beklemishev)

Let $\varphi, \psi \in \mathcal{L}_\Diamond$. *Then:*

$$\mathsf{GL} \vdash \varphi \to \psi$$

$$\Updownarrow$$

$$(\varphi \vdash \psi) \in \mathsf{RC}_1$$

$$\Updownarrow$$

$$\mathsf{PA} \vdash (\varphi \to \psi)^\star \textit{ for any arithmetical realization } (\cdot)^\star$$

$$\Updownarrow$$

$$(\varphi \to \psi) \in \mathsf{PL}(\mathsf{PA})$$

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Even the fragment looks poor, in its polymodal (up to ω) version it suffices for an ordinal notation up to ε_0 and it can perform the main computations of an ordinal analyses of PA and subsystems

Restricted signatures and logics: QRC_1

Restrict $\mathcal{L}_{\Box,\forall}$ to the strictly positive fragment $\mathcal{L}_{\Diamond,\forall}$:

$$\text{Terms} ::= \text{Variables} \mid \text{Constants}$$

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QRC_1 Main result

Theorem (de Almeida Borges, JjJ)

Let $\varphi, \psi \in \mathcal{L}_{\Diamond,\forall}$. *Then:*

$$\varphi \vdash_{\mathsf{QRC}_1} \psi$$
$$\Updownarrow$$
$$\mathsf{PA} \vdash (\varphi \to \psi)^\bullet \textit{ for any arithmetical realization } (\cdot)^\bullet$$
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Theorem (Positive fragment)

Let φ *and* ψ *be* QRC_1 *formulas (no constants) and let* QS *be any logic between* $\mathsf{QK4}$ *and* QGL. *Then* $\varphi \vdash_{\mathsf{QRC}_1} \psi$ *if and only if* $\mathsf{QS} \vdash \varphi \to \psi$.

Four current trends

- Strictly positive fragments of modal logics (Zakharyashev, Wolter, *et al.*)

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- Workshop on Decidable Fragments of First-order Modal Logic, Affiliated workshop of LICS 2022

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Theorem: $\mathsf{QRC}_1 = \{\varphi \vdash \psi \mid \text{for any } (\cdot)^*,\ \mathsf{PA} \vdash (\varphi \vdash \psi)^*\}$

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- Yavorski, add $\Box A \to \Box \forall x A$

Some provable and unprovable statements

$$\Diamond \forall x\, \varphi \vdash \forall x\, \Diamond \varphi$$

$$\forall x\, \Diamond \varphi \nvdash \Diamond \forall x\, \varphi$$

$$\frac{\varphi \vdash \psi[x \leftarrow c]}{\varphi \vdash \forall x\, \psi}$$

x not free in φ and c not in φ nor ψ

Recall that RC_ω allows for ordinal notations up to ε_0 and that it caters Π^0_1 ordinal analyses.

Can be extended to RC_Λ.

Relational models

Kripke models where:

- each world w is a first-order model with a finite domain D
- the domain D is the same for every world
- each constant symbol c and relational symbol S has a denotation at each world
- there is a transitive relation R between worlds
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- the denotation of a relation symbol depends on the world
- we use assignments g : Variables $\to D$ to interpret variables
- we abuse notation and define $g(c) :=$ denotation(c) for all assignments g and constants c

Satisfaction

Let g be a w-assignment.

$$\mathcal{M}, w \Vdash^{g} S(t,u) \iff \langle g(t), g(u)\rangle \in \text{denotation}_w(S)$$

$$\mathcal{M}, w \Vdash^{g} \Diamond\varphi \iff$$
there is a world v such that wRv and $\mathcal{M}, v \Vdash^{g} \varphi$

$$\mathcal{M}, w \Vdash^{g} \forall x\, \varphi \iff$$
for all assignments $h \sim_x g$, we have $\mathcal{M}, w \Vdash^{h} \varphi$

Relational soundness

Theorem (Relational soundness)

If $\varphi \vdash \psi$, then for any model $\mathcal{M}$, world w, and assignment g:

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Countermodels with arbitrarily large domains are needed.

$$\forall x, y\, S(x,x,y) \wedge \forall x, y\, S(x,y,x) \wedge \forall x, y\, S(y,x,x) \vdash \forall x, y, z\, S(x,y,z)$$

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Can be extended to arbitrary n.

Relational completeness

Theorem (Relational completeness)

If $\varphi \nvdash \psi$, then there is a finite model $\mathcal{M}$, a world w, and an assignment g such that:

$$\mathcal{M}, w \Vdash^g \varphi \quad \textit{and} \quad \mathcal{M}, w \nVdash^g \psi.$$

Since QRC_1 has the finite model property (finite number of worlds with finite constant domain), it is decidable.

Arithmetical completeness proof

Theorem (Arithmetical completeness)

$$\mathsf{QRC_1} \supseteq \{\varphi \vdash \psi \mid \textit{for any } (\cdot)^*, \textit{ we have } T \vdash (\varphi \vdash \psi)^*\}$$

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$$\mathsf{QRC}_1 \supseteq \{\varphi \vdash \psi \mid \textit{for any } (\cdot)^*, \ \textit{we have } T \vdash (\varphi \vdash \psi)^*\}$$

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$$(S(x_k))^\bullet := \bigvee_{i \in \mathcal{M}} \Big(\lambda_i \wedge \bigvee_{\langle a \rangle \in S^{\mathcal{M}_i}} \ulcorner a \urcorner = y_k \bmod m \Big)$$

- Prove a Truth Lemma stating (for $i > 0$) that
  - if $i \Vdash^g \chi$ then $T \vdash \lambda_i \to \chi^\bullet[y \leftarrow \ulcorner g(x) \urcorner]$;
  - if $i \nVdash^g \chi$ then $T \vdash \lambda_i \to \neg\chi^\bullet[y \leftarrow \ulcorner g(x) \urcorner]$.
- Conclude (using external reflection) that

$$T \vdash \chi^\bullet[y \leftarrow \ulcorner g(x) \urcorner] \qquad \Leftrightarrow \qquad 1 \Vdash \chi^\bullet[y \leftarrow \ulcorner g(x) \urcorner]$$

for relevant χ whence $\mathsf{PA} \nvdash (\varphi \to \psi)^\bullet[y \leftarrow \ulcorner g(x) \urcorner]$

Main results

Theorem (AdAB, DdJ, JjJ, AV)

Let $\varphi, \psi \in \mathcal{L}_{\Diamond,\forall}$. *Then:*

$$\varphi \vdash_{\mathsf{QRC_1}} \psi$$
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- Recall that PL(HA) is (was) a long-standing open problem

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- But: $\mathrm{HA} \vdash \Box_{\mathrm{HA}} S \leftrightarrow \Box_{\mathrm{HA}} \neg\neg S$ for $S \in \Sigma_1$
- Trick: employ Π_2-conservativity between HA and PA where we have $\mathrm{HA} \vdash \forall A\ (\Box_{\mathrm{HA}} A \to \Box_{\mathrm{PA}} A)$ *for any* A.

Semi-closure

Lemma

- $\mathrm{HA} \vdash \forall S \in \Sigma_1 \ \Box_{\mathrm{HA}} S \leftrightarrow \Box_{\mathrm{HA}} \neg\neg S$
- $\mathrm{HA} \vdash \forall S \in \Sigma_1 \ (\Box_{\mathrm{HA}} \forall x \neg\neg S \leftrightarrow \Box_{\mathrm{HA}} \forall x\, S)$.

The negation of a Π_1 sentence is equivalent to the double negation of a Σ_1 sentence over HA:

Lemma

$$\begin{aligned} \mathrm{HA} \vdash \neg \forall x\, D \quad &\leftrightarrow \quad \neg \forall x \neg\neg D \\ &\leftrightarrow \quad \neg\neg \exists x \neg D \end{aligned} \tag{1}$$

where clearly $\exists x \neg D \in \Sigma_1$.

Lemma

$\mathsf{HA} \vdash \forall A \in \Sigma_2 \ (\Diamond_{\mathsf{HA}} A \leftrightarrow \Diamond_{\mathsf{PA}} A).$

Proof.

In HA, fixing $A \in \Sigma_2$ with $A = \exists x\, P.$ and $S \in \Sigma_1$ so that

$$\neg P \leftrightarrow \neg\neg S. \tag{2}$$

$$\begin{array}{rcll}
\Diamond_{\mathsf{HA}} A & \leftrightarrow & \neg\Box_{\mathsf{HA}} \neg A & \\
 & \leftrightarrow & \neg\Box_{\mathsf{HA}} \neg \exists x\, P & \\
 & \leftrightarrow & \neg\Box_{\mathsf{HA}} \forall x\, \neg P & \\
 & \leftrightarrow & \neg\Box_{\mathsf{HA}} \forall x\, \neg\neg S & \text{by (2)} \\
 & \leftrightarrow & \neg\Box_{\mathsf{HA}} \forall x\, S & \\
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 & \leftrightarrow & \neg\Box_{\mathsf{PA}} \neg\neg \forall x\, S & \\
 & \leftrightarrow & \Diamond_{\mathsf{PA}} \neg \forall x\, S & \\
 & \leftrightarrow & \Diamond_{\mathsf{PA}} \exists x\, \neg S & \\
 & \leftrightarrow & \Diamond_{\mathsf{PA}} A & \text{by (2).}
\end{array}$$

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Theorem

$A \vdash_{\mathsf{RC}_1} B$ if and only if for all realizations $\cdot^*$ we have $\mathsf{HA} \vdash (A \to B)^*$.

Proof.

(Completeness) Assume $A \nvdash_{\mathsf{RC}_1} B$. Embed the extended counter model into arithmetic using the PA Solovay function, which will be our arithmetical interpretation, $\cdot^{\circledast}$.

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This contradicts completeness of RC_1 w.r.t. PA. □

In summary

- There is no quantified provability logic with $\mathcal{L}_{\Box,\forall}$

QRC_1:

- quantified, strictly positive provability logic with $\mathcal{L}_{\Diamond,\forall}$
- decidable
- sound and complete w.r.t. relational semantics (with constant domain models!)
- sound and complete w.r.t. arithmetical semantics
- for all sound r.e. theories extending $\mathsf{I}\Sigma_1$
- Also for HA

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- Strictly positive fragments of modal mu calculus
- Modal mu calculus to capture infinite dynamics in GLP (Reduction Property, Reflexive points in RC models, etc.)

Thank you

Further Reading I

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