My worms, my friends

Joost J. Joosten
University of Barcelona

6th International Wormshop - Bern

31 - 10 - 2023

Happy birthday

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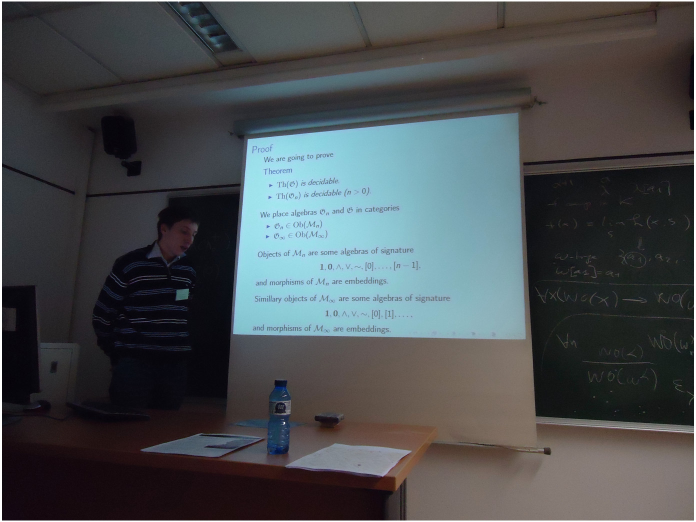
Some photos


Proof
We are going to prove
Theorem
Th(𝔊) is decidable.
Th(𝔊ₙ) is decidable (n > 0).
We place algebras 𝔊ₙ and 𝔊 in categories
Objects of ℳₙ are some algebras of signature
and morphisms of ℳₙ are embeddings.
Similary objects of ℳ∞ are some algebras of signature
and morphisms of ℳ∞ are embeddings.

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- A. Turing's idea: iteration of adding consistency statements
  1938, Systems of logic based on ordinals.
- U. R. Schmerl: adding different kinds of consistency statements
  1978: A fine structure generated by reflection formulas over primitive recursive arithmetic
- L. D. Beklemishev: casting the project in polymodal provability logic
  2004: Provability algebras and proof theoretic ordinals, I

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- And clearly also, elements of a modal logic

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- A happy coincidence?: for worms A, B we can define

$$A <_n B \quad := \mathrm{GLP} \vdash \; B \to \langle n\rangle A$$

and then (Ignatiev, Beklemishev)

$$(\varepsilon_0, \prec) \quad \sim \quad (\mathbb{W}/\sim, <_0)$$

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- is a combinatorial (Hydra like) principle not provable in PA

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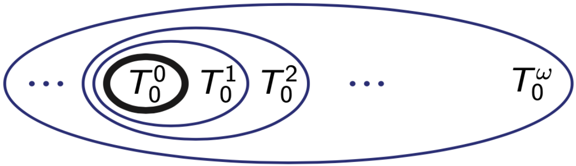
Example



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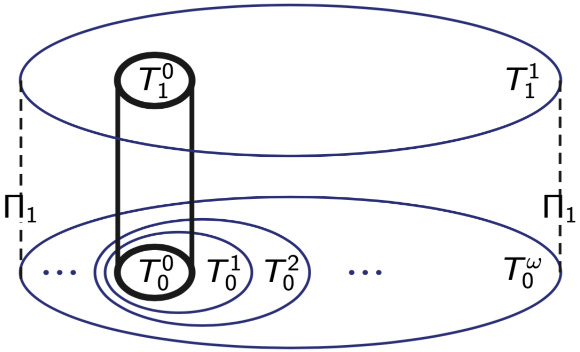

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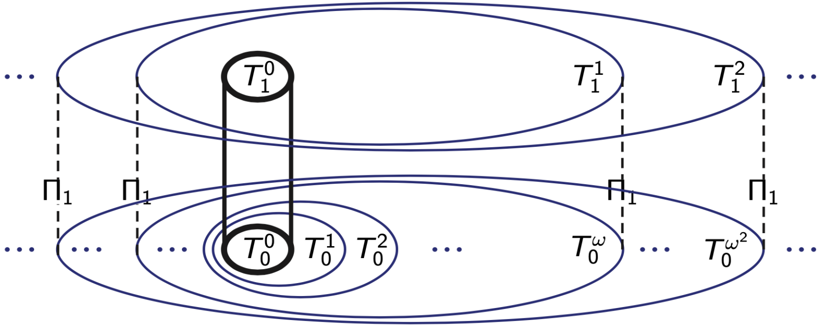

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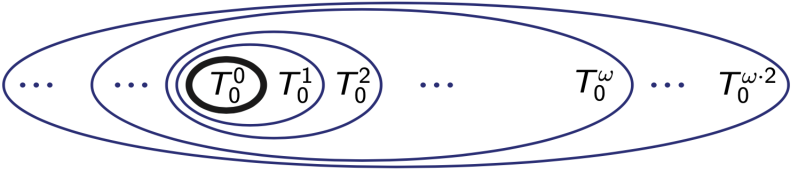

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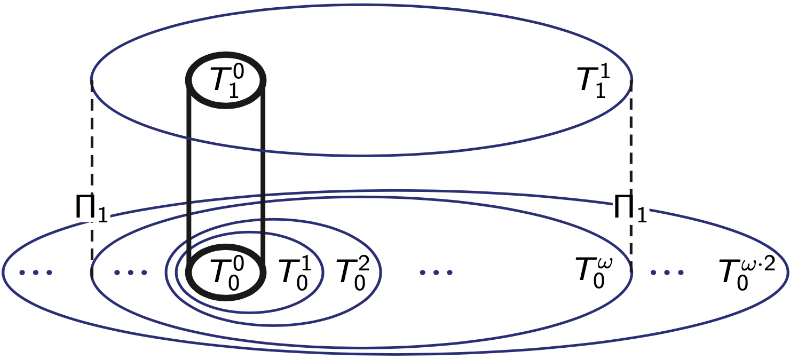

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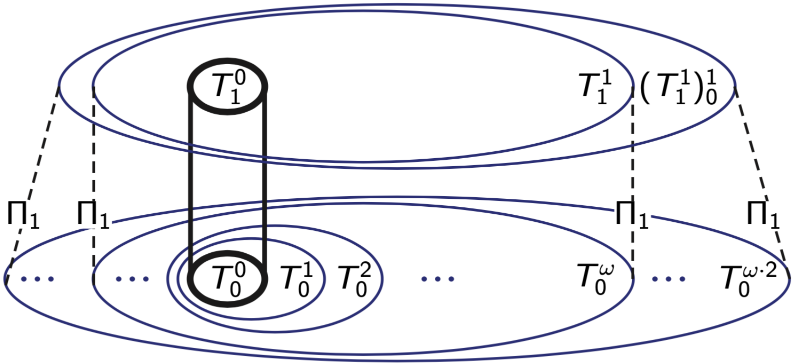

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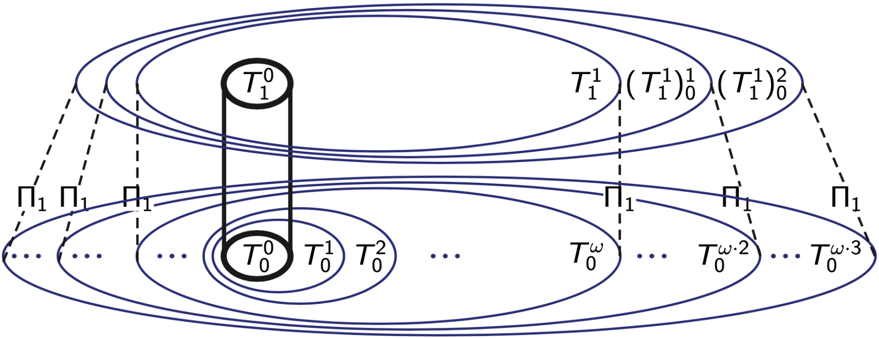

A logic for Turing progressions

- Hermo Reyes, JjJ: A complete calculus (closed fragment) for Turing progressions can be given

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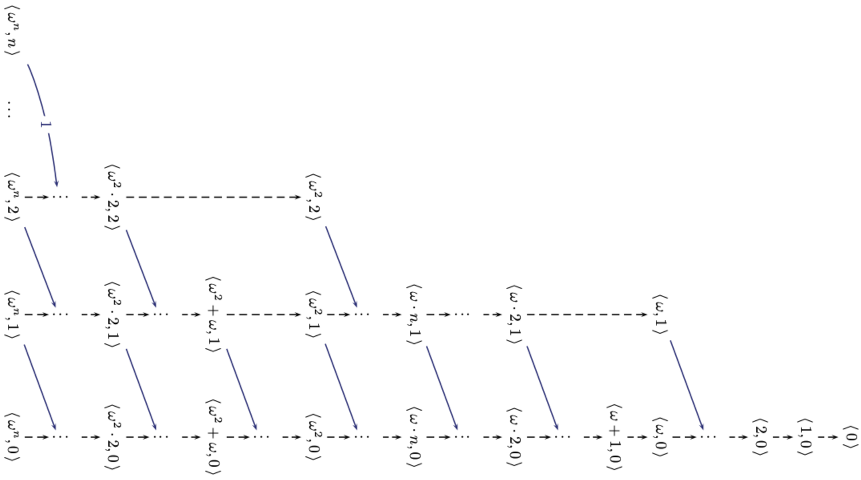
- A picture says more than a honderd words:

Roadmap


$\langle \omega^n, n \rangle$
$\dots$
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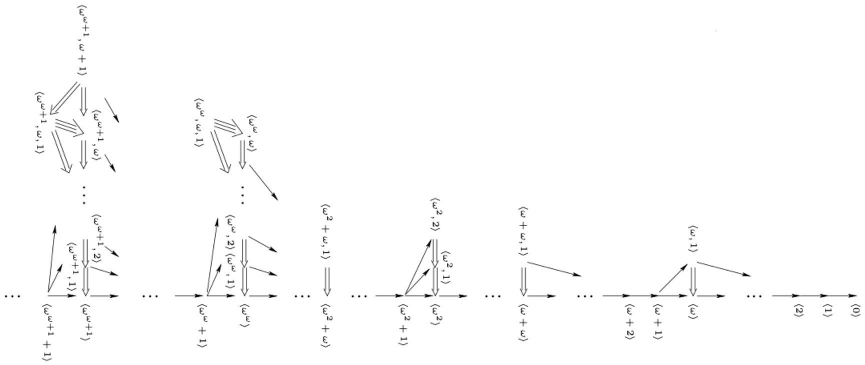
- Worms can denote:
  modal formulas, fragments of arithmetic, ordinals, approximations of Turing progressions, special elements in Ignatiev's model

A universal model



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- For each worm A : $T + A \equiv \bigcup_{n=0}^{\infty} T_n^{o_n(A)}$ (JjJ) whenever T is a Π_1^0 extension of $\mathrm{EA} + \mathrm{supexp}$

  the $o_n(A)$ is the n-order type of A as defined in terms of modal logic later

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- They are not entirely independent:
- $$T_1^1 + T_0^{\omega+1} \equiv T_1^1 + T_0^{\omega\cdot 2}$$
- **Theorem (JjJ)** The Ignatiev sequences exactly correspond to those sub-theories of PA that have a convergent Turing-Taylor expansion

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- The first non-trivial ones longer than omega:

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- **The chopped worm theorem**
  $A \equiv h_\xi(A) \wedge r_\xi(A)$

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- Generalise the GLP_ω sequences iterating exponents/logarithms

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- $e^{\omega^{\beta}}(\xi) = \varphi_{\beta}(\xi)$

Explicit definition for hyperations (DfD, JjJ)

Don't read:

- **Definition** Let $e(\xi) = -1 + \omega^{\xi}$. Then, we define the *hyperexponential* $e^{\zeta}\xi$ by the following recursion:
- $e^0\xi = \xi$
- $e^{\xi}0 = 0$
- $e^1 = e$
- $e^{\omega^\rho + \xi} = e^{\omega^\rho} e^{\xi}$, where $\xi < \omega^\rho + \xi$
- $e^{\omega^\rho}(\xi + 1) = \lim\limits_{\zeta \to \omega^\rho} e^{\zeta}(e^{\omega^\rho}(\xi) + 1)$, provided $\rho > 0$
- $e^{\omega^\rho}\xi = \lim\limits_{\zeta \to \xi} e^{\omega^\rho}\zeta$ for $\xi \in \mathrm{Lim}$, $\rho > 0$.

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- I^{α}

$$
\begin{tikzcd}
\mathbb{W} \arrow[r, "A \mapsto 0A"] \arrow[d, "o"] & \mathbb{W} \arrow[d, "o"] \\
\mathsf{On} \arrow[r, "+1"] & \mathsf{On}
\end{tikzcd}
$$

$$\begin{tikzcd} \mathbb{W} \times \mathbb{W} \arrow[r, "\langle A,B\rangle \mapsto A0B"] \arrow[d, "o"] & \mathbb{W} \arrow[d, "o"] \\ \mathsf{On} \times \mathsf{On} \arrow[r, "\langle\alpha,\beta\rangle\mapsto\beta+1+\alpha"] & \mathsf{On} \end{tikzcd}$$

$$\begin{tikzcd} \mathbb{W} \arrow[r, "1\uparrow"] \arrow[d, "o"] & \mathbb{W}_1 \arrow[d, "o"] \\ \mathsf{On} \arrow[r, "e"] & \mathsf{On} \end{tikzcd}$$

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- The latter naturally suggests a generalisation to the transfinite

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Predicativity through reflection

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$$\mathsf{PA} \;\equiv\; \mathsf{EA} + \{\langle 1\rangle_{\mathsf{EA}}\top, \langle 2\rangle_{\mathsf{EA}}\top, \langle 3\rangle_{\mathsf{EA}}\top, \ldots\}$$

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- As before, link between consistency and reflection

$\mathsf{ATR}_0 \equiv \mathsf{ECA}_0 + \mathtt{Pred\text{-}O\text{-}Cons}(\mathsf{ECA}_0) \equiv \mathsf{ECA}_0 + \mathtt{Pred\text{-}O\text{-}RFN}_{\mathbf{\Pi}^1_2}(\mathrm{ACA}).$

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- Employing the fact that $\Box_T$ is Σ^0_1-complete

Münchhausen provability

- Main idea:

$$[\zeta]^{\Lambda}_{T}\phi \quad :\Leftrightarrow \quad \Box_T\phi \ \vee \ \exists\psi\,\exists\,\xi{<}\zeta\ \big(\langle\xi\rangle^{\Lambda}_{T}\psi \ \wedge \ \Box_T(\langle\xi\rangle^{\Lambda}_{T}\psi \to \phi)\big).$$

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- Runs in phase with Turing jumps and thus seems to allow for fine structures

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- DfD, JjJ, Pakhomov, Papafillipou, Weierman: stronger versions of EWD

Denoting theories

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RC: Axioms and rules

Dashkov, Beklemishev

Formulas built from propositional variables, conjunctions and consistency statements denote theories

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- de Almeida Borges, JjJ: Worm calculus
  There is a reflection calculus based solely on worms ($A \vdash B$) deciding all GLP $\vdash A \to B$ theorems

QRC_1: Axioms and rules

Strictly positive formulas φ, ψ, χ using only universal quantifiers, conjunctions and consistency statements:

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t free for x in φ and ψ $\qquad$ c not in φ nor ψ

Some provable and unprovable statements

$$\Diamond \forall x\, \varphi \vdash \forall x\, \Diamond \varphi$$

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x not free in φ and c not in φ nor ψ

Arithmetical semantics

The arithmetical realizations $(\cdot)^*$ for $\mathcal{L}_{\Diamond,\forall}$:

$$\text{formulas in } \mathcal{L}_{\Diamond,\forall} \to \text{axiomatisations of c.e. theories in } \mathcal{L}_{\mathsf{PA}}$$
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Arithmetical soundness and completeness

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- Embed such models in arithmetic using the Solovay sentences λ_i ...

Heyting arithmetic

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- Various other proof theoretical notions also suffer from pathological orders

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- They moreover showed how techniques à la Schmerl/Beklemishev could be employed to prove:

$$|\boldsymbol{R}^{\alpha}_{\Pi^1_1}(\mathsf{ACA}_0)|_{\mathsf{ACA}_0} = \alpha \text{ and } |\boldsymbol{R}^{\alpha}_{\Pi^1_1}(\mathsf{ACA}_0)|_{\mathsf{WO}} = \varepsilon_\alpha$$

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- and then proves a reversal over RCA_0 of hyperations preserving well-foundedness to $\mathbf{\Pi}^1_3{-}\omega\mathsf{RFN}(\Pi^1_1{-}BI)$

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- Over $\mathsf{KP}_0\omega$ foundation can be expressed as iterated reflection:

$$\mathsf{KP}\omega \equiv \mathsf{RFN}^{\varepsilon_{\boldsymbol{\Omega}_{n+1}}}_{\Pi^0_2}\mathsf{KP}_0\omega$$

Long live worms