Eliminating cuts from non-nested reflection calculi

Joost J. Joosten
Cosimo Perini Brogi

University of Barcelona / CRM
IMT School for Advanced Studies Lucca

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The question

Can a reflection calculus have well-behaved sequent proof systems?

$A \Rightarrow B$ with one formula on each side.

No contexts, labels, nested sequents, or hypersequents in the main calculus.

Target: cut-free calculi, syntactic conservativity over GL, uniform interpolation, and complexity.

Reflection calculi: RC_1 and QRC_1

$$A ::= p \mid \top \mid A \wedge B \mid \Diamond A \qquad A ::= S(\vec{t}) \mid \top \mid A \wedge B \mid \Diamond A \mid \forall x A$$

RC_1: positive core

$$\frac{}{A \vdash A}\ \mathrm{id} \qquad \frac{}{A \vdash \top}\ \top \qquad \frac{A \vdash B \qquad A \vdash C}{A \vdash B \wedge C}\ \wedge R$$

$$\frac{}{A \wedge B \vdash A}\ \wedge\mathcal{L}_1 \qquad \frac{}{A \wedge B \vdash B}\ \wedge\mathcal{L}_2 \qquad \frac{A \vdash B}{\Diamond A \vdash \Diamond B}\ \mathrm{Nec}$$

$$\frac{}{\Diamond\Diamond A \vdash \Diamond A}\ 4 \qquad \frac{A \vdash B \qquad B \vdash C}{A \vdash C}\ \mathrm{Cut}$$

QRC_1: relational atoms + universal quantifier

$$\frac{A[x \leftarrow t] \vdash B}{\forall x A \vdash B}\ \forall L \qquad \frac{A \vdash B[x \leftarrow y]}{A \vdash \forall x B}\ \forall R$$

$$\frac{A \vdash B}{A[x \leftarrow t] \vdash B[x \leftarrow t]}\ \mathrm{Inst} \qquad \frac{A[x \leftarrow c] \vdash B[x \leftarrow c]}{A \vdash B}\ \mathrm{CE}$$

Side conditions: t free for x where required; y an eigenvariable not free in the conclusion; c fresh for A, B in CE.

Three presentations of modal transitivity

Axioms are restricted to atoms: $\dfrac{}{p \vdash p}\ \mathrm{id}$ or $\dfrac{}{R(t_0, \ldots, t_n) \vdash R(t_0, \ldots, t_n)}\ \mathrm{id}$

Analytic	**Explicit**	**Reduced**
$\mathrm{G1RC1}$	$\mathrm{G1K4}^+$	$\mathrm{G1K4}^+_*$
$\dfrac{A \Rightarrow \Diamond B}{\Diamond A \Rightarrow \Diamond B}\ 4\mathrm{Nec}$	$\dfrac{\Diamond A \Rightarrow B}{\Diamond\Diamond A \Rightarrow B}\ 4L$	$\dfrac{A \Rightarrow \Diamond\Diamond B}{A \Rightarrow \Diamond B}\ 4R$
	$\dfrac{A \Rightarrow \Diamond\Diamond B}{A \Rightarrow \Diamond B}\ 4R$	No $4L$.

Idea

$\mathrm{G1K4}^+$ though non-analytic is tame enough to allow Cut elimination; $\mathrm{G1RC1}$ is the right place to search.

$$\frac{A \Rightarrow C \qquad C \Rightarrow B}{A \Rightarrow B}\ \mathrm{Cut}$$

Theorem (core cut admissibility)

If $\mathrm{G1RC1} \vdash A \Rightarrow C$ *and* $\mathrm{G1RC1} \vdash C \Rightarrow B$*, then* $\mathrm{G1RC1} \vdash A \Rightarrow B$*.*

The induction measure is not cut rank:

$$h_l + h_r$$

Direct Cut in the explicit calculus

$$\frac{A \Rightarrow C \qquad C \Rightarrow B}{A \Rightarrow B}\ \mathrm{Cut}$$

Theorem (core cut admissibility)

If $\mathrm{G1K4}^+ \vdash A \Rightarrow C$ *and* $\mathrm{G1K4}^+ \vdash C \Rightarrow B$, *then* $\mathrm{G1K4}^+ \vdash A \Rightarrow B$.

The induction measure is not cut rank:

$$\langle h_l, h_r \rangle \quad \text{lexicographic, with } h_l \text{ primary.}$$

Critical modal reductions may increase the right height, but only after the left height has dropped.

One critical modal reduction

$$\dfrac{\dfrac{A \Rightarrow \Diamond\Diamond\Diamond C}{A \Rightarrow \Diamond\Diamond C}\,4R \qquad \dfrac{\Diamond C \Rightarrow B}{\Diamond\Diamond C \Rightarrow B}\,4L}{A \Rightarrow B}\,\mathrm{Cut}$$

Replace it by:

$$A \Rightarrow \Diamond\Diamond\Diamond C \qquad \Diamond\Diamond\Diamond C \Rightarrow B$$

where the second derivation is obtained by applying $4L$ once more.

The new measure is

$$\langle h_l - 1, h_r + 1\rangle <_{\mathrm{lex}} \langle h_l, h_r\rangle.$$

Quantifiers as a module

The quantified extension does not change the modal reductions.

We use the standard proof-theoretic variable discipline:

$$\alpha\text{-conversion of bound variables} \quad \Rightarrow \quad \text{substitution} \quad \Rightarrow \quad \text{eigenvariable renaming.}$$

The only genuinely new principal cut is:

$$\frac{A \Rightarrow C[x \leftarrow y]}{A \Rightarrow \forall x C}\,\forall R \qquad \frac{C[x \leftarrow t] \Rightarrow B}{\forall x C \Rightarrow B}\,\forall L$$

Substitute t for the eigenvariable y in the left premise, then cut the premises.

Conservativity over GL: use the dual presentation

Dashkov's theorem:

$$A \vdash_{\mathsf{RC}_1} B \quad \Longleftrightarrow \quad \mathsf{GL} \vdash A \to B \qquad (A, B \in \mathcal{L}^{s+}).$$

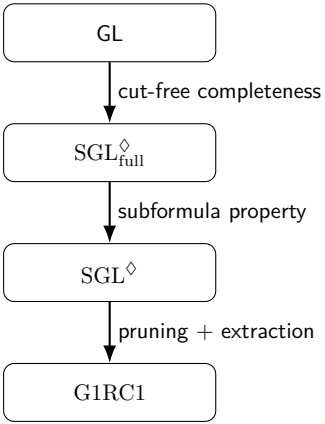
Rather than translate positive formulas into $\Box$-formulas, take $\Diamond$ as primitive:

$$\Diamond A \equiv \neg\Box\neg A.$$

$$\frac{A \Rightarrow \Sigma, \Diamond\Sigma, \Diamond A}{\Omega, \Diamond A \Rightarrow \Diamond\Sigma, \Delta}\ \Diamond\text{-Löb}$$

This is the De Morgan-dual of the ordinary $\Box$-Löb rule for cut-free multiset sequents.

Positive restriction



If the end-sequent is strictly positive, analyticity prevents any formula with $\neg$ from appearing.

Positive pruning and extraction

Theorem

If $\mathrm{SGL}^{\Diamond} \vdash \Gamma \Rightarrow \Delta$, *then some occurrence* $D \in \Delta$ *satisfies*

$$\mathrm{SGL}^{\Diamond} \vdash \Gamma \Rightarrow D \qquad \textit{and} \qquad \mathrm{G1RC1} \vdash \bigwedge \Gamma \Rightarrow D.$$

The modal-depth witness excludes the diagonal succedent:

$$\mathrm{SGL}^{\Diamond} \nvdash A \Rightarrow \Diamond A.$$

In the Löb case, the selected premise occurrence is either

$$S \in \Sigma \quad \rightsquigarrow \quad \mathrm{Nec}, \qquad \Diamond S \in \Diamond \Sigma \quad \rightsquigarrow \quad 4\mathrm{Nec}.$$

Uniform interpolation for the calculi

Fix retained relation symbols Λ and retained free parameters Y.
The projection is defined structurally on formulas:

$$A \longmapsto A\restriction_{\Lambda,Y} .$$

Private atomic information is collapsed to $\top$; quantifiers are handled by extending the local parameter set.

Theorem (calculus-level uniform post-interpolation)

If

$$\mathrm{G1QRC1} \vdash A \Rightarrow B \quad \textit{and} \quad B \textit{ uses only } (\Lambda, Y),$$

then

$$\mathrm{G1QRC1} \vdash A\restriction_{\Lambda,Y} \Rightarrow B.$$

For $\mathrm{G1RC1}$, this is the nullary case: uniform interpolation for RC_1.

Proof idea: induction on cut-free derivations; only $\forall L$ uses admissible Cut.

Proof search

The analytic calculus $\mathrm{G1QRC1}$ gives a memoised backwards search relation

$$D(A, B) \quad \text{“there is a cut-free derivation of } A \Rightarrow B\text{”}.$$

For the propositional core, states are pairs of subformulas:

$$(C, D) \in \mathrm{Sub}(A) \times \mathrm{Sub}(B).$$

For QRC_1, the only new finite data are term instantiations for $\forall L$ and eigenparameters for $\forall R$.

The cut-free proof itself is the certificate.

Complexity

Fragment	Upper bound	Matching lower bound
RC_1	deterministic polynomial time	–
QRC_1, fixed effective signature	NP	–
QRC_1 with one binary relation	NP	NP-hard
modality-free bounded left $\forall$ depth d	$O(N^{d+2})$	–

Hardness is by reduction from Complete Partition / pseudoachromatic number.

It already uses closed, constant-free formulas of modal depth one over a single binary relation symbol.

What the proof theory buys

1. System design: three equivalent formula-to-formula calculi.
2. Modular Cut: direct for $\mathrm{G1K4}^{+}$, transferred to the analytic calculi.
3. Dashkov syntactically: GL $\to \mathrm{SGL}^{\Diamond} \to \mathrm{G1RC1} \to \mathsf{RC}_1$.
4. Uniform interpolation: structural projection inside the calculi.
5. Complexity: analytic proof search gives small certificates.
6. Counter model extraction: bounds on the model size.
   $\mathsf{ht}(W, R) \leq d$, $\max_{w \in W} |D(w)| \leq K^{2^d}$, and $|W| \leq \left(2N^2K^q\right)^{2^d}$. In particular, $|W| = N^{\mathcal{O}((q+2)2^d)}$.

The strict positivity is doing the structural work.

References

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Backup: the three proof systems again

$$\begin{aligned} \mathsf{RC}_1 &\equiv \mathrm{G1RC1} \equiv \mathrm{G1K4}^+ \equiv \mathrm{G1K4}^+_*, \\ \mathsf{QRC}_1 &\equiv \mathrm{G1QRC1} \equiv \mathrm{G1QK4}^+ \equiv \mathrm{G1QK4}^+_*. \end{aligned}$$

The equivalences are effective derivation transformations, not merely semantic completeness results.

Backup: why $4L$ disappears

$$\frac{\Diamond A \Rightarrow C}{\Diamond\Diamond A \Rightarrow C}\ \text{admissible } 4L$$

The proof is by induction on $\langle h(\Pi), |C| \rangle$.

For $C = \Diamond C_0$, use Nec and $4\mathrm{Nec}$; for $C = C_1 \wedge C_2$, use right inversion and the secondary induction.

Backup: finite countermodels and model size

Failed proof search can also be read semantically.

Countermodel extraction

If $\mathrm{G1QRC1} \nvdash A \Rightarrow B$, construct a finite expanding-domain Kripke model with

$$\mathcal{M}, w \models A \qquad \text{and} \qquad \mathcal{M}, w \not\models B.$$

Quantitative control:

$$\mathrm{height} \leq d_{\Diamond}(A), \qquad |D(w)| \leq K, \qquad |W| \leq S_d.$$