Sherlock Holmes vs. the jewel thief

Sherlock Holmes and Watson are chasing a jewel thief through downtown London. . . but lose him just as they arrive at the train station!

The two watch perplexed as they see several trains leaving the station in all directions.

But Sherlock is not shaken!

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Watson: Brilliant! So we'll buy the ticket we find most unexpected and take the journey. But which one shall we take?

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Holmes: Simple. The A train was carrying a police squadron to take them to a training camp... and the thief would never risk getting on a train with so many policemen! Watson, buy a ticket to the A train!

Watson: Holmes, you never cease to amaze me!

The thief in Europe

Thus Sherlock and Watson board the next A train but it's too late. . . the thief has left England! Now, Sherlock and Watson must chase him through Europe!

But he can go anywhere. . . or can he?

The thief in Europe

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Watson And we know he bought a ticket from Amsterdam to Berlin so that is perfect proof that he will not stay in Holland. . . so he must have gone there! But where will he go next?

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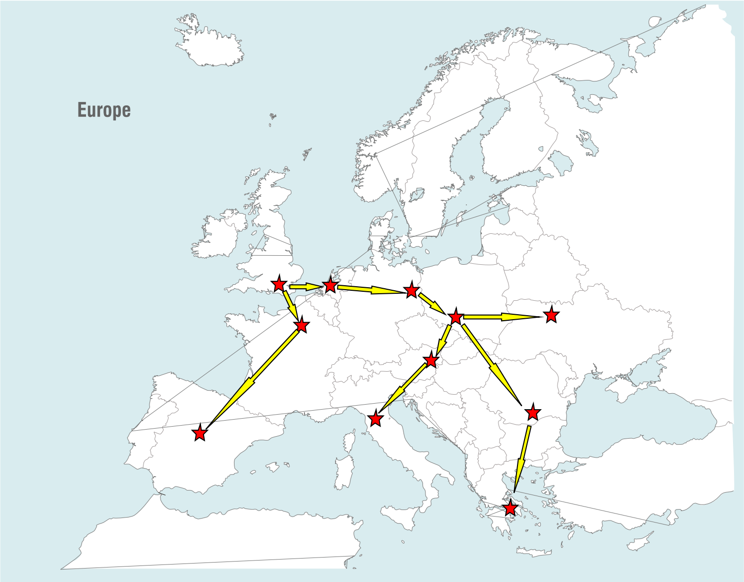
Watson Holmes, here is a map of the main train routes...

The train routes



Chasing the thief

Thus Sherlock and Watson set off to chase the thief through Europe! Their journey takes them to Amsterdam... to Berlin... and finally to Opole! But they can't find the thief!

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Where may he be?

This time he outsmarted Holmes. . .

The thief's whereabouts

Solovay's theorem

This situation illustrates the proof of Solovay's theorem, which we will see today:

Theorem (Solovay)

If GL $\nvdash \phi$ *there is an arithmetic interpretation* f *such that* PA $\nvdash f(\phi)$.

Proof sketch

Theorem

GL *is complete for its arithmetic interpretation.*

Proof.

1. Assume ϕ is a consistent GL formula.
2. Pick a GL-model $\mathfrak{M}$ satisfying ϕ.
3. For each world w of $\mathfrak{M}$ build a formula $\mathtt{Thief}(w)$ satisfying Solovay's conditions.
4. Define $f(p) = \bigvee_{w \in V(p)} \mathtt{Thief}(w)$.

$\square$

The thief in a Kripke model

We have a GL-model $\mathfrak{M} = \langle W, \succ, V \rangle$ such that

- $W = \{1, 2, \ldots, n\}$
- $\succ$ is a partial order
- 1 is the root (maximum element)

We add an “imaginary root” 0.

The thief in a Kripke model

On day 0, we begin at the world 0, and will follow a path

$$w_0, w_1, \ldots w_n.$$

The formula $\texttt{Thief}(v)$ asserts "$w_n = v$"
The thief will go to the world u on day $i + 1$ if

1. $w_i \succ u$
2. i codes a proof of $\neg\texttt{Thief}(u)$.

The journey finishes at w_n, from which there is nowhere he can go.

What path will the thief follow?

Solovay paths

A *Solovay path* is a sequence $\vec{w} = \langle w_0, \ldots, w_N \rangle$ such that

- $\vec{w} = \langle w_0, \ldots, w_N \rangle$ is a sequence
- $w_0 = 0$
- for all $n < N$, $w_n \neq w_{n+1}$ if and only if $w_n \succ w_{n+1}$ and n codes a proof of

  "The last element of $\vec{w}$ is not w_{n+1}"

  and
- for all $n > N$ and $u \prec w_N$, n does not code a proof of

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Can we write a formula $\texttt{Thief}(w)$ expressing "there is a Solovay path ending in w"?

Solovay paths a bit more formally

$\mathtt{Thief}(w)$ should be equivalent to:
There exists a sequence $\vec{w} = \langle w_0, \ldots, w_N \rangle$ *such that*

- $w_0 = 0$ *and* $w_N = w$
- *for all* $n < N$, $w_n \neq w_{n+1}$ *if and only if* $w_n \succ w_{n+1}$ *and* n *codes a proof of* $\neg\mathtt{Thief}(w_{n+1})$ *and*
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OK, but how do we find such a formula?

Solovay paths even more formally

Let us rewrite $\mathtt{Thief}(w)$:
There exists a path $\vec{w} = \langle w_0, \ldots, w_N \rangle$ *such that*

- $w_0 = 0$ *and* $w_N = w$
- *for all* $n < N$, $w_n \neq w_{n+1}$ *if and only if* $w_n \succ w_{n+1}$ *and*

$$\mathtt{prv}_{\mathsf{PA}}\left(\ulcorner \neg \mathtt{Thief}(\overline{w_{n+1}}) \urcorner, n\right)$$

and

- *for all* $n > N$ *and* $u \prec w_N$,

$$\neg \mathtt{prv}_{\mathsf{PA}}\left(\ulcorner \neg \mathtt{Thief}(\overline{u}) \urcorner, n\right).$$

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This only depends on $\ulcorner \texttt{Thief}(x) \urcorner$ and not $\texttt{Thief}(x)$ itself!

Solovay paths even more formally

Define a predicate $\psi(\overline{\ulcorner \texttt{Thief}(x) \urcorner}, w)$ by:
$\exists y \exists N\, \texttt{seq}(y) \wedge \texttt{Last}(\texttt{y}, \texttt{N})$ and

- $\texttt{entry}(y, 0, 0) \wedge \texttt{entry}(y, N, w)$,
- for all $n < N$, $w_n \neq w_{n+1}$ if and only if $w_n \succ w_{n+1}$ and

$$\texttt{prv}_{\mathsf{PA}}\left(\overline{\ulcorner \neg \texttt{Thief}(w_{n+1}) \urcorner}, n\right)$$

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- for all $n > N$ and $u \prec w_N$,

$$\neg \mathtt{prv}_{\mathsf{PA}}\big(\overline{\ulcorner \neg \mathtt{Thief}(u) \urcorner}, n\big).$$

Fixpoint theorem: there is a formula $\mathtt{Thief}(w)$ such that for all w,

$$\mathsf{PA} \vdash \mathtt{Thief}(w) \leftrightarrow \psi(\overline{\ulcorner \mathtt{Thief}(x) \urcorner}, w).$$

Solovay properties

- PA $\vdash \neg \bigvee_{w \neq v} \texttt{Thief}(w) \wedge \texttt{Thief}(v)$

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- if $w \neq 0$, PA $+ \mathtt{Thief}(w)$ proves $\Box_{\mathrm{PA}} \bigvee_{v \prec w} \mathtt{Thief}(v)$
- $\mathbb{N} \models \mathtt{Thief}(0)$

Correctness

Proposition

Define $f(p) = \bigvee_{w \in V(p)} \mathtt{Thief}(w)$

- *If* $w \neq 0$, $\mathfrak{M}, w \models \psi \Leftrightarrow \mathsf{PA} + \mathtt{Thief}(w) \vdash f(\psi)$

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- $\mathsf{PA} + \texttt{Thief}(0) \vdash \Diamond_{\mathsf{PA}} \texttt{Thief}(1)$

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- $\mathsf{PA} + \mathtt{Thief}(0) \vdash \Diamond_{\mathsf{PA}} \mathtt{Thief}(1)$

Since $\mathbb{N} \models \mathsf{PA} + \mathtt{Thief}(0)$, we conclude that $f(\phi)$ is consistent.

We have shown that $\mathsf{GL} \nvdash \phi$ implies that there is an interpretation f such that $\mathsf{PA} \nvdash f(\phi)$, as we wanted.