Provability Logics and Applications
Day 3
Polymodal logics

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ESSLLI Tutorial, Opole

Polymodal Gödel-Löb

GLP$_\omega$: Contains one modality $[n]$ for each $n < \omega$.

Axioms:

$$
\begin{array}{rr}
[n](\varphi \to \psi) \to ([n]\varphi \to [n]\psi) & (n < \omega) \\
[n]([n]\varphi \to \varphi) \to [n]\varphi & (n < \omega) \\
[n]\varphi \to [m]\varphi & (n < m < \omega) \\
\langle n \rangle\varphi \to [m]\langle n \rangle\varphi & (n < m < \omega)
\end{array}
$$

Rules: Modus ponens and necessitation for all modalities

Introduced by Japaridze in 1988.

Arithmetic interpretations

Recall: Π_n formulas are arithmetic formulas of the form

$$\forall x_0 \exists x_1 \forall x_2 \ldots Q_n x_n \phi$$

If $f : \mathbb{P} \to \mathrm{L}_{\mathrm{PA}}$, we extend f to all L_ω:

- f commutes with Booleans
- $f([0]\phi) = \Box_T \phi := \mathtt{Prv}_T(\overline{\ulcorner \phi \urcorner})$
- for $n > 0$, $f([n]\phi)$ means
  "Provable in T together with the set of all true Π_n sentences."

Interpretation of $[n]$

Theorem

For all $n < \omega$ there is a formula $\mathtt{True}_n(x)$ such that $\mathbb{N} \models \mathtt{True}_n(\overline{k})$ if and only if k codes a true Π_n-sentence.

With this, we can formalize n-provability:

$$[n]_T(x) := \exists y \big(\mathtt{True}_n(y) \wedge \Box_T(y \to x)\big).$$

Note: if T is elementarily presented then $[n]_T(x)$ is Σ_{n+1}.

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Theorem

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2. $[n]([n]\phi \to \phi) \to [n]\phi$
3. $[n]\phi \to [n+1]\phi$

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2. $[n]([n]\phi \to \phi) \to [n]\phi$
3. $[n]\phi \to [n+1]\phi$
4. $\langle n\rangle\phi \to [n+1]\langle n\rangle\phi$

□

Kripke semantics

Frames:

$$\langle W, \langle >_n \rangle_{n<\omega} \rangle$$

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Valid iff

$$w >_n v \text{ and } w >_{n+1} u \Rightarrow u >_n v$$

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Even GLP_2 has no non-trivial Kripke models.

Topologies

Topological spaces:

- A set X, a family of subsets $\mathcal{T} \subseteq \mathcal{P}(X)$
- $\varnothing, X \in \mathcal{T}$
- $U, V \in \mathcal{T} \Rightarrow U \cap V \in \mathcal{T}$
- $\mathcal{O} \subseteq \mathcal{T} \rightarrow \bigcup \mathcal{O} \in \mathcal{T}$
- elements of $\mathcal{T}$ are open.

The derived-set operator

Neighborhoods of x: $x \in U \in \mathcal{T}$

Limit points of A: x such that if U is a neighborhood of x then there is $y \in A \cap U$ different from x.

Set of limit points of A: dA

Scattered spaces: Every non-empty set has an isolated point.

Spaces:

$$\langle X, \langle \mathcal{T}_n \rangle_{n<\omega} \rangle$$

Write d_n for the limit point operator on $\mathcal{T}_n$.

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$[n]\varphi \to [n+1]\varphi$: Valid iff $\mathcal{T}_n \subseteq \mathcal{T}_{n+1}$

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$[n]\varphi \to [n+1]\varphi$: Valid iff $\mathcal{T}_n \subseteq \mathcal{T}_{n+1}$

$\langle n \rangle\varphi \to [n+1]\langle n \rangle\varphi$: Valid iff

$$A \subseteq X \Rightarrow d_n A \in \mathcal{T}_{n+1}$$

Topological completeness

Beklemishev, Gabelaia: GLP is complete for the class of GLP-spaces

The proof uses non-constructive methods.

Blass: It is consistent with ZFC that the canonical ordinal spaces for GLP_2 are all trivial

Beklemishev: It is also consistent with ZFC that GLP_2 is complete for its canonical ordinal spaces

Bagaria More generally, for all n it is consistent with ZFC that GLP_n has non-trivial canonical ordinal spaces but GLP_{n+1} does not.