Provability Logics and Applications Day 1 Provability as modality

David Fernández Duque¹ and Joost J. Joosten²

1: Universidad de Sevilla;

2: Universitat de Barcelona

Monday 13-08-2012 ESSLLI Tutorial, Opole

Formalized Modus Ponens and Deduction Sigma completeness of PA Löb's theorem

э.

From now on: no distinction between A, $\lceil A \rceil$, $\overline{\lceil A \rceil}$, etc.

Formalized Modus Ponens and Deduction Sigma completeness of PA Löb's theorem

▲□ → ▲圖 → ▲ 国 → ▲ 国 → →

- From now on: no distinction between A, $\lceil A \rceil$, $\overline{\lceil A \rceil}$, etc.
- If the context allows us to

・ロン ・回 と ・ ヨ と ・ ヨ と …

- From now on: no distinction between A, $\lceil A \rceil$, $\overline{\lceil A \rceil}$, etc.
- If the context allows us to
- If we allow ourselves to...

- From now on: no distinction between A, $\lceil A \rceil$, $\overline{\lceil A \rceil}$, etc.
- If the context allows us to
- If we allow ourselves to...
- Easy to see:

$$\mathbb{N}\models \mathtt{Prv}_{\mathrm{PA}}(A o B)\wedge \mathtt{Prv}_{\mathrm{PA}}(A) o \mathtt{Prv}_{\mathrm{PA}}(B)$$

・ロト ・回ト ・ヨト ・ヨト

- From now on: no distinction between A, $\lceil A \rceil$, $\overline{\lceil A \rceil}$, etc.
- If the context allows us to
- If we allow ourselves to...
- Easy to see:

$$\mathbb{N}\models \mathtt{Prv}_{\mathrm{PA}}(A o B)\wedge \mathtt{Prv}_{\mathrm{PA}}(A) o \mathtt{Prv}_{\mathrm{PA}}(B)$$

イロト イヨト イヨト イヨト

æ

For example via Hilbert style implementation of Prv_{PA}.

- From now on: no distinction between A, $\lceil A \rceil$, $\overline{\lceil A \rceil}$, etc.
- If the context allows us to
- If we allow ourselves to...
- Easy to see:

$$\mathbb{N}\models \mathtt{Prv}_{\mathrm{PA}}(A o B)\wedge \mathtt{Prv}_{\mathrm{PA}}(A) o \mathtt{Prv}_{\mathrm{PA}}(B)$$

イロト イヨト イヨト イヨト

- For example via Hilbert style implementation of Prv_{PA}.
- ▶ We can *construct* $\operatorname{prv}_{\operatorname{PA}}(z, B)$ given $\operatorname{prv}_{\operatorname{PA}}(x, A \to B)$ and $\operatorname{prv}_{\operatorname{PA}}(y, A)$

- From now on: no distinction between A, $\lceil A \rceil$, $\overline{\lceil A \rceil}$, etc.
- If the context allows us to
- If we allow ourselves to...
- Easy to see:

$$\mathbb{N}\models \mathtt{Prv}_{\mathrm{PA}}(A o B)\wedge \mathtt{Prv}_{\mathrm{PA}}(A) o \mathtt{Prv}_{\mathrm{PA}}(B)$$

- For example via Hilbert style implementation of Prv_{PA}.
- ▶ We can *construct* $prv_{PA}(z, B)$ given $prv_{PA}(x, A \rightarrow B)$ and $prv_{PA}(y, A)$

Thus:

$$\mathrm{PA} \vdash \mathtt{Prv}_{\mathrm{PA}}(\mathcal{A} \to \mathcal{B}) \land \mathtt{Prv}_{\mathrm{PA}}(\mathcal{A}) \to \mathtt{Prv}_{\mathrm{PA}}(\mathcal{B})$$

・ロト ・回ト ・ヨト ・ヨト

- From now on: no distinction between A, $\lceil A \rceil$, $\overline{\lceil A \rceil}$, etc.
- If the context allows us to
- If we allow ourselves to...
- Easy to see:

$$\mathbb{N}\models \mathtt{Prv}_{\mathrm{PA}}(A o B)\wedge \mathtt{Prv}_{\mathrm{PA}}(A) o \mathtt{Prv}_{\mathrm{PA}}(B)$$

- For example via Hilbert style implementation of Prv_{PA}.
- ▶ We can *construct* $prv_{PA}(z, B)$ given $prv_{PA}(x, A \rightarrow B)$ and $prv_{PA}(y, A)$

Thus:

$$\mathrm{PA} \vdash \mathtt{Prv}_{\mathrm{PA}}(\mathcal{A}
ightarrow \mathcal{B}) \land \mathtt{Prv}_{\mathrm{PA}}(\mathcal{A})
ightarrow \mathtt{Prv}_{\mathrm{PA}}(\mathcal{B})$$

<ロ> (日) (日) (日) (日) (日)

Provable/Formalized Modus Ponens

Formalized Modus Ponens and Deduction Sigma completeness of PA Löb's theorem

æ

Provable/Formalized Modus Ponens

イロト イヨト イヨト イヨト

- Provable/Formalized Modus Ponens
- ▶ We also have a formalized version of the Deduction Theorem

- Provable/Formalized Modus Ponens
- ▶ We also have a formalized version of the Deduction Theorem

Theorem

$$\mathbb{N}\models \mathtt{Prv}_{\mathrm{PA}+\mathcal{A}}(B) \leftrightarrow \mathtt{Prv}_{\mathrm{PA}}(\mathcal{A}
ightarrow B)$$

・ロン ・回 と ・ ヨ と ・ ヨ と

- Provable/Formalized Modus Ponens
- ▶ We also have a formalized version of the Deduction Theorem

Theorem

$$\mathbb{N}\models \mathtt{Prv}_{\mathrm{PA}+\mathcal{A}}(B) \; \leftrightarrow \; \mathtt{Prv}_{\mathrm{PA}}(\mathcal{A}
ightarrow B)$$

- Provable/Formalized Modus Ponens
- We also have a formalized version of the Deduction Theorem

Theorem

$$\mathbb{N}\models \mathtt{Prv}_{\mathrm{PA}+\mathcal{A}}(B) \; \leftrightarrow \; \mathtt{Prv}_{\mathrm{PA}}(\mathcal{A}
ightarrow B)$$

イロン 不同と 不同と 不同と

æ

 \blacktriangleright \rightarrow follows from induction on the length of a proof;

- Provable/Formalized Modus Ponens
- ▶ We also have a formalized version of the Deduction Theorem

Theorem

$$\mathbb{N}\models \mathtt{Prv}_{\mathrm{PA}+\mathcal{A}}(B) \; \leftrightarrow \; \mathtt{Prv}_{\mathrm{PA}}(\mathcal{A}
ightarrow B)$$

イロト イヨト イヨト イヨト

- \blacktriangleright \rightarrow follows from induction on the length of a proof;
- Hilbert style calculus: only deal with Modus Ponens.

- Provable/Formalized Modus Ponens
- ▶ We also have a formalized version of the Deduction Theorem

Theorem

$$\mathbb{N}\models \mathtt{Prv}_{\mathrm{PA}+\mathcal{A}}(B) \; \leftrightarrow \; \mathtt{Prv}_{\mathrm{PA}}(\mathcal{A}
ightarrow \mathcal{B})$$

<ロ> <同> <同> <同> < 同> < 同>

- \blacktriangleright \rightarrow follows from induction on the length of a proof;
- Hilbert style calculus: only deal with Modus Ponens.
- ▶ Note, PA can perform this induction!

- Provable/Formalized Modus Ponens
- ▶ We also have a formalized version of the Deduction Theorem

Theorem

$$\mathbb{N}\models \mathtt{Prv}_{\mathrm{PA}+\mathcal{A}}(B) \; \leftrightarrow \; \mathtt{Prv}_{\mathrm{PA}}(\mathcal{A}
ightarrow \mathcal{B})$$

- \blacktriangleright \rightarrow follows from induction on the length of a proof;
- Hilbert style calculus: only deal with Modus Ponens.
- Note, PA can perform this induction!
- ► So:

$$\mathrm{PA} \vdash \mathtt{Prv}_{\mathrm{PA}+\mathcal{A}}(\mathcal{B}) \iff \mathtt{Prv}_{\mathrm{PA}}(\mathcal{A} \to \mathcal{B})$$

イロト イヨト イヨト イヨト

Formalized Modus Ponens and Deduction Sigma completeness of $\ensuremath{\operatorname{PA}}$ Löb's theorem

æ

► Gödel I: PA is incomplete

David Fernández Duque¹ and Joost J. Joosten² Provability as modality

< □ > < □ > < □ > < □ > < □ > .

- ► Gödel I: PA is incomplete
- ▶ That is, there is some true sentence π with $PA \nvDash \pi$

イロン 不同と 不同と 不同と

- ► Gödel I: PA is incomplete
- ▶ That is, there is some true sentence π with $PA \nvDash \pi$
- We have seen such a π :

イロン イヨン イヨン イヨン

- ► Gödel I: PA is incomplete
- ▶ That is, there is some true sentence π with $PA \nvDash \pi$
- We have seen such a π : The Gödel sentence λ

イロト イヨト イヨト イヨト

- ► Gödel I: PA is incomplete
- ▶ That is, there is some true sentence π with $PA \nvDash \pi$
- We have seen such a π : The Gödel sentence λ
- Note that $\lambda \in \Pi_1$

イロト イヨト イヨト イヨト

- ► Gödel I: PA is incomplete
- ▶ That is, there is some true sentence π with $PA \nvDash \pi$
- We have seen such a π : The Gödel sentence λ
- Note that $\lambda \in \Pi_1$
- Thus: PA is Π_1 -incomplete

イロト イヨト イヨト イヨト

- Gödel I: PA is incomplete
- ▶ That is, there is some true sentence π with $PA \nvDash \pi$
- We have seen such a π : The Gödel sentence λ
- Note that $\lambda \in \Pi_1$
- Thus: PA is Π_1 -incomplete
- We shall now see that this is optimal

<ロ> <同> <同> <同> < 同> < 同>

- Gödel I: PA is incomplete
- ▶ That is, there is some true sentence π with $PA \nvDash \pi$
- We have seen such a π : The Gödel sentence λ
- Note that $\lambda \in \Pi_1$
- Thus: PA is Π_1 -incomplete
- We shall now see that this is optimal
- Theorem PA is Σ₁-complete

イロト イヨト イヨト イヨト

- Gödel I: PA is incomplete
- ▶ That is, there is some true sentence π with $PA \nvDash \pi$
- We have seen such a π : The Gödel sentence λ
- Note that $\lambda \in \Pi_1$
- Thus: PA is Π_1 -incomplete
- We shall now see that this is optimal
- Theorem PA is Σ_1 -complete
- $\blacktriangleright \mathbb{N} \models \sigma \quad \Rightarrow \quad \mathrm{PA} \vdash \sigma \text{ for } \sigma \in \Sigma_1$

Formalized Modus Ponens and Deduction Sigma completeness of PA Löb's theorem

▲□▶ ▲圖▶ ▲≣▶ ▲≣▶ 三重 - のへぐ

$$\blacktriangleright \mathbb{N} \models \sigma \quad \Rightarrow \quad \mathrm{PA} \vdash \sigma \text{ for } \sigma \in \Sigma_1$$

Formalized Modus Ponens and Deduction Sigma completeness of $\ensuremath{\operatorname{PA}}$ Löb's theorem

- $\blacktriangleright \mathbb{N} \models \sigma \quad \Rightarrow \quad \mathrm{PA} \vdash \sigma \text{ for } \sigma \in \Sigma_1$
- \blacktriangleright **Proof**: by induction on the complexity of σ

イロン イヨン イヨン イヨン

- $\blacktriangleright \mathbb{N} \models \sigma \quad \Rightarrow \quad \mathrm{PA} \vdash \sigma \text{ for } \sigma \in \Sigma_1$
- Proof: by induction on the complexity of σ
- True atomic sentences can all be proved in PA

イロン イヨン イヨン イヨン

- $\blacktriangleright \mathbb{N} \models \sigma \quad \Rightarrow \quad \mathrm{PA} \vdash \sigma \text{ for } \sigma \in \Sigma_1$
- **Proof**: by induction on the complexity of σ
- True atomic sentences can all be proved in PA

•
$$t_1 = t_2$$
 and $t_1 < t_2$

- $\blacktriangleright \mathbb{N} \models \sigma \quad \Rightarrow \quad \mathrm{PA} \vdash \sigma \text{ for } \sigma \in \Sigma_1$
- **Proof**: by induction on the complexity of σ
- True atomic sentences can all be proved in PA
- $t_1 = t_2$ and $t_1 < t_2$
- ▶ By induction on the complexity of t_1 and sufficient for $t_1 = \overline{n}$

イロン イヨン イヨン イヨン

- $\blacktriangleright \mathbb{N} \models \sigma \quad \Rightarrow \quad \mathrm{PA} \vdash \sigma \text{ for } \sigma \in \Sigma_1$
- **Proof**: by induction on the complexity of σ
- True atomic sentences can all be proved in PA
- $t_1 = t_2$ and $t_1 < t_2$
- ▶ By induction on the complexity of t_1 and sufficient for $t_1 = \overline{n}$ *n* times

イロン イヨン イヨン イヨン

2

• For example, in $a + b = S \underbrace{S \dots S}_{0} 0$

- $\blacktriangleright \mathbb{N} \models \sigma \quad \Rightarrow \quad \mathrm{PA} \vdash \sigma \text{ for } \sigma \in \Sigma_1$
- **Proof**: by induction on the complexity of σ
- True atomic sentences can all be proved in PA
- $t_1 = t_2$ and $t_1 < t_2$
- ▶ By induction on the complexity of t_1 and sufficient for $t_1 = \overline{n}$ *n* times
- For example, in $a + b = S \underbrace{\overline{S \dots S}}_{0} 0$

▶
$$b = 0$$
, then $a + 0 = a$ and by induction PA $\vdash a = S \underbrace{S \dots S}_{S \dots S} 0$;

イロン イヨン イヨン イヨン

- $\blacktriangleright \mathbb{N} \models \sigma \quad \Rightarrow \quad \mathrm{PA} \vdash \sigma \text{ for } \sigma \in \Sigma_1$
- Proof: by induction on the complexity of σ
- True atomic sentences can all be proved in PA
- $t_1 = t_2$ and $t_1 < t_2$
- ▶ By induction on the complexity of t_1 and sufficient for $t_1 = \overline{n}$ *n* times
- For example, in $a + b = S \underbrace{\overline{S \dots S}}_{0} 0$

•
$$b = 0$$
, then $a + 0 = a$ and by induction $PA \vdash a = S \underbrace{5 \dots 5}_{n \text{ times}} 0$;
and using an axiom: $PA \vdash a + b = S \underbrace{5 \dots 5}_{n \dots 5} 0$

イロン イヨン イヨン イヨン

- $\blacktriangleright \mathbb{N} \models \sigma \quad \Rightarrow \quad \mathrm{PA} \vdash \sigma \text{ for } \sigma \in \Sigma_1$
- **Proof**: by induction on the complexity of σ
- True atomic sentences can all be proved in PA
- $t_1 = t_2$ and $t_1 < t_2$
- ▶ By induction on the complexity of t_1 and sufficient for $t_1 = \overline{n}$ *n* times
- For example, in $a + b = S \underbrace{\overline{S \dots S}}_{0} 0$

•
$$b = 0$$
, then $a + 0 = a$ and by induction $PA \vdash a = S \underbrace{S \dots S}_{n \text{ times}} 0$;
and using an axiom: $PA \vdash a + b = S \underbrace{S \dots S}_{n \text{ times}} 0$
• $b = Sb'$, then $a + Sb' = S(a + b')$ whence by IH
 $a + b' = \underbrace{S \dots S}_{n \text{ times}} 0$

・ロト ・回ト ・ヨト ・ヨト

- $\blacktriangleright \mathbb{N} \models \sigma \quad \Rightarrow \quad \mathrm{PA} \vdash \sigma \text{ for } \sigma \in \Sigma_1$
- **Proof**: by induction on the complexity of σ
- True atomic sentences can all be proved in PA
- $t_1 = t_2$ and $t_1 < t_2$
- ▶ By induction on the complexity of t_1 and sufficient for $t_1 = \overline{n}$ *n* times
- For example, in $a + b = S \underbrace{\overline{S \dots S}}_{0} 0$

•
$$b = 0$$
, then $a + 0 = a$ and by induction $PA \vdash a = S \underbrace{S \dots S}_{n \text{ times}} 0$;
and using an axiom: $PA \vdash a + b = S \underbrace{S \dots S}_{n \text{ times}} 0$
• $b = Sb'$, then $a + Sb' = S(a + b')$ whence by IH
 $a \text{ times}$
 $PA \vdash a + b' = \underbrace{S \dots S}_{n \text{ times}} 0$
and using an axiom: $PA \vdash a + b = S \underbrace{S \dots S}_{n \text{ times}} 0$

Formalized Modus Ponens and Deduction Sigma completeness of PA Löb's theorem

・ロン ・回 と ・ヨン ・ヨン

3

▶ True atomic sentences can all be proved in PA

・ロト ・回ト ・ヨト ・ヨト

- True atomic sentences can all be proved in PA
- We have seen one simple case

イロト イヨト イヨト イヨト

- True atomic sentences can all be proved in PA
- We have seen one simple case
- Many more cases but equally simple

イロト イヨト イヨト イヨト

- True atomic sentences can all be proved in PA
- We have seen one simple case
- Many more cases but equally simple
- ▶ Next step: bounded quantification $\forall x < y \ \psi(x)$

- True atomic sentences can all be proved in PA
- We have seen one simple case
- Many more cases but equally simple
- Next step: bounded quantification $\forall x < y \ \psi(x)$
- For each natural number y

$$\mathrm{PA} \vdash \forall x < \overline{y} \ \psi(x) \ \leftrightarrow \ \underbrace{\psi(0) \land \ldots \land \psi(\overline{y})}^{y+1 \text{ conjuncts}}$$

<ロ> <同> <同> <同> < 同> < 同>

- True atomic sentences can all be proved in PA
- We have seen one simple case
- Many more cases but equally simple
- Next step: bounded quantification $\forall x < y \ \psi(x)$
- For each natural number y

$$\mathrm{PA} \vdash \forall x < \overline{y} \ \psi(x) \ \leftrightarrow \ \underbrace{\psi(0) \land \ldots \land \psi(\overline{y})}^{y+1 \text{ conjuncts}}$$

- A 🗇 N - A 🖻 N - A 🖻 N

Thus, we can apply our IH

- True atomic sentences can all be proved in PA
- We have seen one simple case
- Many more cases but equally simple
- ▶ Next step: bounded quantification $\forall x < y \ \psi(x)$
- For each natural number y

$$\mathrm{PA} \vdash \forall x < \overline{y} \ \psi(x) \ \leftrightarrow \ \widetilde{\psi(0) \land \ldots \land \psi(\overline{y})}$$

- A 同 ト - A 三 ト - A 三 ト

- Thus, we can apply our IH
- Likewise for $\exists x < \overline{y} \psi(x)$

Formalized Modus Ponens and Deduction Sigma completeness of PA Löb's theorem

イロン イヨン イヨン イヨン

æ

Boolean connectives: by an easy call to the IH

David Fernández Duque¹ and Joost J. Joosten² Provability as modality

イロト イヨト イヨト イヨト

- Boolean connectives: by an easy call to the IH
- Unbounded existential quantification: $\exists x \ \psi(x)$

<ロ> (日) (日) (日) (日) (日)

- Boolean connectives: by an easy call to the IH
- Unbounded existential quantification: $\exists x \ \psi(x)$
- also directly from the IH

イロト イヨト イヨト イヨト

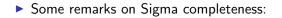
- Boolean connectives: by an easy call to the IH
- Unbounded existential quantification: $\exists x \ \psi(x)$
- also directly from the IH
- This finishes the proof

- Boolean connectives: by an easy call to the IH
- Unbounded existential quantification: $\exists x \ \psi(x)$
- also directly from the IH
- This finishes the proof

$$\mathbb{N} \models \sigma \Rightarrow \operatorname{PA} \vdash \sigma \text{ for } \sigma \in \Sigma_1$$

イロト イヨト イヨト イヨト

Formalized Modus Ponens and Deduction Sigma completeness of $\ensuremath{\operatorname{PA}}$ Löb's theorem



イロト イヨト イヨト イヨト

- Some remarks on Sigma completeness:
- Goldbach's conjecture: each even number above two is the sum of two prime numbers

イロト イヨト イヨト イヨト

- Some remarks on Sigma completeness:
- Goldbach's conjecture: each even number above two is the sum of two prime numbers
- This is a Π_1 statement

- Some remarks on Sigma completeness:
- Goldbach's conjecture: each even number above two is the sum of two prime numbers
- This is a Π_1 statement
- Thus: If Goldbach's conjecture is independent of PA, then it is true

- ∢ ⊒ ⊳

- Some remarks on Sigma completeness:
- Goldbach's conjecture: each even number above two is the sum of two prime numbers
- This is a Π_1 statement
- Thus: If Goldbach's conjecture is independent of PA, then it is true

イロト イヨト イヨト イヨト

▶ **Theorem:** If $PA \vdash \varphi$, then $PA \vdash Prv_{PA}(\varphi)$

- Some remarks on Sigma completeness:
- Goldbach's conjecture: each even number above two is the sum of two prime numbers
- This is a Π_1 statement
- Thus: If Goldbach's conjecture is independent of PA, then it is true
- Theorem: If $PA \vdash \varphi$, then $PA \vdash Prv_{PA}(\varphi)$
- ▶ **Proof:** Remember, representing "provability in PA " implies

p is the code of a PA proof of $\varphi \iff \mathbb{N} \models \mathtt{prv}_{\mathrm{PA}}(p, \varphi)$

・ロト ・回ト ・ヨト ・ヨト

- Some remarks on Sigma completeness:
- Goldbach's conjecture: each even number above two is the sum of two prime numbers
- This is a Π_1 statement
- Thus: If Goldbach's conjecture is independent of PA, then it is true
- Theorem: If $PA \vdash \varphi$, then $PA \vdash Prv_{PA}(\varphi)$
- ▶ Proof: Remember, representing "provability in PA " implies

p is the code of a PA proof of $\varphi \iff \mathbb{N} \models \mathtt{prv}_{\mathrm{PA}}(p, \varphi)$

• As
$$Prv_{PA}(\varphi) \in \Sigma_1$$
 we have

$$\mathrm{PA} \vdash \varphi \Rightarrow \mathbb{N} \models \mathrm{Prv}_{\mathrm{PA}}(\varphi) \Rightarrow \mathrm{PA} \vdash \mathrm{Prv}_{\mathrm{PA}}(\varphi)$$

・ロト ・回ト ・ヨト ・ヨト

3

► Sigma completeness: $\mathbb{N} \models \sigma \implies \mathrm{PA} \vdash \sigma$ for $\sigma \in \Sigma_1$

・ロト ・回ト ・ヨト ・ヨト

- ► Sigma completeness: $\mathbb{N} \models \sigma \Rightarrow PA \vdash \sigma$ for $\sigma \in \Sigma_1$
- The proof gives us slightly more:

イロン イヨン イヨン イヨン

- ► Sigma completeness: $\mathbb{N} \models \sigma \Rightarrow PA \vdash \sigma$ for $\sigma \in \Sigma_1$
- The proof gives us slightly more:
- ► Theorem $PA \vdash ``N \models \sigma \Rightarrow PA \vdash \sigma''$ for $\sigma \in \Sigma_1$

イロン イヨン イヨン イヨン

- ► Sigma completeness: $\mathbb{N} \models \sigma \Rightarrow PA \vdash \sigma$ for $\sigma \in \Sigma_1$
- The proof gives us slightly more:
- ► Theorem $PA \vdash "\mathbb{N} \models \sigma \Rightarrow PA \vdash \sigma"$ for $\sigma \in \Sigma_1$
- That is: $PA \vdash \sigma \rightarrow Prv_T(\sigma)$ for $\sigma \in \Sigma_1$

・ロト ・回ト ・ヨト ・ヨト

- ► Sigma completeness: $\mathbb{N} \models \sigma \Rightarrow PA \vdash \sigma$ for $\sigma \in \Sigma_1$
- The proof gives us slightly more:
- ► Theorem $PA \vdash "\mathbb{N} \models \sigma \Rightarrow PA \vdash \sigma"$ for $\sigma \in \Sigma_1$
- That is: $PA \vdash \sigma \rightarrow Prv_T(\sigma)$ for $\sigma \in \Sigma_1$
- Proof: formalizing exactly the previous proof of Sigma-completeness in PA

イロト イヨト イヨト イヨト

- ► Sigma completeness: $\mathbb{N} \models \sigma \Rightarrow PA \vdash \sigma$ for $\sigma \in \Sigma_1$
- The proof gives us slightly more:
- ► Theorem $PA \vdash "\mathbb{N} \models \sigma \Rightarrow PA \vdash \sigma"$ for $\sigma \in \Sigma_1$
- That is: $PA \vdash \sigma \rightarrow Prv_T(\sigma)$ for $\sigma \in \Sigma_1$
- Proof: formalizing exactly the previous proof of Sigma-completeness in PA
- We have a sufficient amount of induction

- ► Sigma completeness: $\mathbb{N} \models \sigma \Rightarrow PA \vdash \sigma$ for $\sigma \in \Sigma_1$
- The proof gives us slightly more:
- ► Theorem $PA \vdash "\mathbb{N} \models \sigma \Rightarrow PA \vdash \sigma"$ for $\sigma \in \Sigma_1$
- That is: $PA \vdash \sigma \rightarrow Prv_{\mathcal{T}}(\sigma)$ for $\sigma \in \Sigma_1$
- Proof: formalizing exactly the previous proof of Sigma-completeness in PA
- We have a sufficient amount of induction
- Note that we can do bounded quantification as we have the totality of exponentiation

・ロト ・回ト ・ヨト ・ヨト

- ► Sigma completeness: $\mathbb{N} \models \sigma \Rightarrow PA \vdash \sigma$ for $\sigma \in \Sigma_1$
- The proof gives us slightly more:
- ► Theorem $PA \vdash "\mathbb{N} \models \sigma \Rightarrow PA \vdash \sigma"$ for $\sigma \in \Sigma_1$
- That is: $PA \vdash \sigma \rightarrow Prv_{\mathcal{T}}(\sigma)$ for $\sigma \in \Sigma_1$
- Proof: formalizing exactly the previous proof of Sigma-completeness in PA
- We have a sufficient amount of induction
- Note that we can do bounded quantification as we have the totality of exponentiation

イロト イヨト イヨト イヨト

2

▶ Corollary: $PA \vdash Prv_{PA}(\varphi) \rightarrow Prv_{PA}(Prv_{PA}(\varphi))$

Formalized Modus Ponens and Deduction Sigma completeness of PA Löb's theorem

▲口 → ▲圖 → ▲ 国 → ▲ 国 → □

æ

Provable Σ₁-completeness:

```
\mathrm{PA} \vdash \mathtt{Prv}_{\mathrm{PA}}(\varphi) \rightarrow \mathtt{Prv}_{\mathrm{PA}}(\mathtt{Prv}_{\mathrm{PA}}(\varphi))
```

Provable Σ₁-completeness:

```
\mathrm{PA} \vdash \mathtt{Prv}_{\mathrm{PA}}(\varphi) \to \mathtt{Prv}_{\mathrm{PA}}(\mathtt{Prv}_{\mathrm{PA}}(\varphi))
```

<ロ> (日) (日) (日) (日) (日)

æ

 Corollary: Gödel II: If a theory is consistent, it will not prove its own consistency.

Provable Σ₁-completeness:

```
\mathrm{PA} \vdash \mathtt{Prv}_{\mathrm{PA}}(\varphi) \to \mathtt{Prv}_{\mathrm{PA}}(\mathtt{Prv}_{\mathrm{PA}}(\varphi))
```

<ロ> (日) (日) (日) (日) (日)

- Corollary: Gödel II: If a theory is consistent, it will not prove its own consistency.
- **Proof** We see that $PA \vdash \lambda \leftrightarrow \neg Prv_{PA}(\bot)$

Provable Σ₁-completeness:

```
\mathrm{PA} \vdash \mathtt{Prv}_{\mathrm{PA}}(\varphi) \rightarrow \mathtt{Prv}_{\mathrm{PA}}(\mathtt{Prv}_{\mathrm{PA}}(\varphi))
```

イロン イヨン イヨン イヨン

2

- Corollary: Gödel II: If a theory is consistent, it will not prove its own consistency.
- **Proof** We see that $PA \vdash \lambda \leftrightarrow \neg Prv_{PA}(\bot)$
- Clearly, inside PA we have $\neg Prv_{PA}(\lambda) \rightarrow \neg Prv_{PA}(\bot)$.

Provable Σ₁-completeness:

```
\mathrm{PA} \vdash \mathtt{Prv}_{\mathrm{PA}}(\varphi) \rightarrow \mathtt{Prv}_{\mathrm{PA}}(\mathtt{Prv}_{\mathrm{PA}}(\varphi))
```

- Corollary: Gödel II: If a theory is consistent, it will not prove its own consistency.
- **Proof** We see that $PA \vdash \lambda \leftrightarrow \neg Prv_{PA}(\bot)$
- Clearly, inside PA we have $\neg Prv_{PA}(\lambda) \rightarrow \neg Prv_{PA}(\perp)$.
- ▶ For the other direction reason in PA and assume $\neg Prv_{PA}(\bot)$

イロン イヨン イヨン イヨン

Provable Σ₁-completeness:

```
\mathrm{PA} \vdash \mathtt{Prv}_{\mathrm{PA}}(\varphi) \rightarrow \mathtt{Prv}_{\mathrm{PA}}(\mathtt{Prv}_{\mathrm{PA}}(\varphi))
```

- Corollary: Gödel II: If a theory is consistent, it will not prove its own consistency.
- **Proof** We see that $PA \vdash \lambda \leftrightarrow \neg Prv_{PA}(\bot)$
- Clearly, inside PA we have $\neg Prv_{PA}(\lambda) \rightarrow \neg Prv_{PA}(\perp)$.
- ▶ For the other direction reason in PA and assume $\neg Prv_{PA}(\bot)$

イロト イヨト イヨト イヨト

• Moreover, for a contradiction assume $Prv_{PA}(\lambda)$

Provable Σ₁-completeness:

```
\mathrm{PA} \vdash \mathtt{Prv}_{\mathrm{PA}}(\varphi) \rightarrow \mathtt{Prv}_{\mathrm{PA}}(\mathtt{Prv}_{\mathrm{PA}}(\varphi))
```

- Corollary: Gödel II: If a theory is consistent, it will not prove its own consistency.
- **Proof** We see that $PA \vdash \lambda \leftrightarrow \neg Prv_{PA}(\bot)$
- Clearly, inside PA we have $\neg Prv_{PA}(\lambda) \rightarrow \neg Prv_{PA}(\bot)$.
- ▶ For the other direction reason in PA and assume $\neg Prv_{PA}(\bot)$

(ロ) (同) (E) (E) (E)

- Moreover, for a contradiction assume $Prv_{PA}(\lambda)$
- ▶ By provable Σ_1 completeness: $Prv_{PA}(Prv_{PA}(\lambda))$, that is $Prv_{PA}(\neg\lambda)$

Formalized Modus Ponens and Deduction Sigma completeness of ${\rm PA}$ Löb's theorem

イロン イヨン イヨン イヨン

æ

▶ Theorem If $PA \vdash Prv_{PA}(\varphi) \rightarrow \varphi$, then $PA \vdash \varphi$

- ▶ Theorem If $PA \vdash Prv_{PA}(\varphi) \rightarrow \varphi$, then $PA \vdash \varphi$
- PA is as modest about it's own correctness as it could possibly be

・ロト ・回 ト ・ヨト ・ヨトー

イロト イヨト イヨト イヨト

- ▶ Theorem If $PA \vdash Prv_{PA}(\varphi) \rightarrow \varphi$, then $PA \vdash \varphi$
- PA is as modest about it's own correctness as it could possibly be
- **Proof:** By contraposition supposing $PA \nvDash \varphi$

イロン イヨン イヨン イヨン

- ▶ Theorem If $PA \vdash Prv_{PA}(\varphi) \rightarrow \varphi$, then $PA \vdash \varphi$
- PA is as modest about it's own correctness as it could possibly be
- ▶ **Proof:** By contraposition supposing $PA \nvDash \varphi$
- Thus $PA + \neg \varphi$ is consistent

- ▶ Theorem If $PA \vdash Prv_{PA}(\varphi) \rightarrow \varphi$, then $PA \vdash \varphi$
- PA is as modest about it's own correctness as it could possibly be
- **Proof:** By contraposition supposing $PA \nvDash \varphi$
- Thus $PA + \neg \varphi$ is consistent
- ▶ By Gödel 2 for $PA + \neg \varphi$ we get

 $\mathrm{PA} + \neg \varphi \nvDash \mathtt{Con}_{\mathrm{PA} + \neg \varphi}$

イロン イヨン イヨン イヨン

3

- ▶ Theorem If $PA \vdash Prv_{PA}(\varphi) \rightarrow \varphi$, then $PA \vdash \varphi$
- PA is as modest about it's own correctness as it could possibly be
- **Proof:** By contraposition supposing $PA \nvDash \varphi$
- Thus $PA + \neg \varphi$ is consistent
- By Gödel 2 for $PA + \neg \varphi$ we get

 $PA + \neg \varphi \nvDash Con_{PA + \neg \varphi}$

By deduction (and the formalized version)

$$\mathrm{PA} \nvDash \neg \varphi \to \mathtt{Con}_{\mathrm{PA}}(\neg \varphi)$$

イロン イヨン イヨン イヨン

2

- ▶ Theorem If $PA \vdash Prv_{PA}(\varphi) \rightarrow \varphi$, then $PA \vdash \varphi$
- PA is as modest about it's own correctness as it could possibly be
- **Proof:** By contraposition supposing $PA \nvDash \varphi$
- Thus $PA + \neg \varphi$ is consistent
- By Gödel 2 for $PA + \neg \varphi$ we get

 $PA + \neg \varphi \nvDash Con_{PA + \neg \varphi}$

By deduction (and the formalized version)

$$\mathrm{PA} \nvDash \neg \varphi \to \mathtt{Con}_{\mathrm{PA}}(\neg \varphi)$$

イロト イヨト イヨト イヨト

• And $Con_{PA}(\neg \varphi)$ is just $\neg Prv_{PA}(\varphi)$

Syntax of modal logics Various modal logics

イロン イヨン イヨン イヨン

æ

$\blacktriangleright \operatorname{PA} \vdash \operatorname{Prv}_{\operatorname{PA}}(A \to B) \land \operatorname{Prv}_{\operatorname{PA}}(A) \to \operatorname{Prv}_{\operatorname{PA}}(B)$

David Fernández Duque¹ and Joost J. Joosten² Provability as modality

 $\blacktriangleright \ \mathrm{PA} \vdash \texttt{Prv}_{\mathrm{PA}}(\mathcal{A} \to \mathcal{B}) \land \texttt{Prv}_{\mathrm{PA}}(\mathcal{A}) \to \texttt{Prv}_{\mathrm{PA}}(\mathcal{B})$

 Note, this holds for any (possibly non-standard) formulas A and B

イロト イヨト イヨト イヨト

- $\blacktriangleright \operatorname{PA} \vdash \operatorname{\texttt{Prv}}_{\operatorname{PA}}(\mathcal{A} \to \mathcal{B}) \land \operatorname{\texttt{Prv}}_{\operatorname{PA}}(\mathcal{A}) \to \operatorname{\texttt{Prv}}_{\operatorname{PA}}(\mathcal{B})$
- Note, this holds for any (possibly non-standard) formulas A and B

イロト イヨト イヨト イヨト

æ

▶ We would like to collect *all* such principles

- $\blacktriangleright \ \mathrm{PA} \vdash \texttt{Prv}_{\mathrm{PA}}(\mathcal{A} \to \mathcal{B}) \land \texttt{Prv}_{\mathrm{PA}}(\mathcal{A}) \to \texttt{Prv}_{\mathrm{PA}}(\mathcal{B})$
- Note, this holds for any (possibly non-standard) formulas A and B

<ロ> (日) (日) (日) (日) (日)

- We would like to collect all such principles
- If possible

- $\blacktriangleright \operatorname{PA} \vdash \operatorname{\tt Prv}_{\operatorname{PA}}(A \to B) \land \operatorname{\tt Prv}_{\operatorname{PA}}(A) \to \operatorname{\tt Prv}_{\operatorname{PA}}(B)$
- Note, this holds for any (possibly non-standard) formulas A and B
- ▶ We would like to collect *all* such principles
- If possible
- We have to find a suitable signature where to collect such principles

<ロ> (日) (日) (日) (日) (日)

- $\blacktriangleright \operatorname{PA} \vdash \operatorname{Prv}_{\operatorname{PA}}(A \to B) \land \operatorname{Prv}_{\operatorname{PA}}(A) \to \operatorname{Prv}_{\operatorname{PA}}(B)$
- Note, this holds for any (possibly non-standard) formulas A and B
- ▶ We would like to collect *all* such principles
- If possible
- We have to find a suitable signature where to collect such principles

イロト イヨト イヨト イヨト

æ

Propositional modal logics

Syntax of modal logics Various modal logics

・ロン ・聞と ・ほと ・ほと

æ

Language of propositional modal logic:

Syntax of modal logics Various modal logics

- 4 回 2 - 4 □ 2 - 4 □

- Language of propositional modal logic:
 - ► countable set of propositional variables P;

Syntax of modal logics Various modal logics

| 4 回 2 4 U = 2 4 U =

- Language of propositional modal logic:
 - ► countable set of propositional variables P;
 - Two logical constants \top and \bot .

Syntax of modal logics Various modal logics

- Language of propositional modal logic:
 - ► countable set of propositional variables P;
 - Two logical constants \top and \bot .
- Operators of propositional modal logic:

- 4 回 ト - 4 回 ト - 4 回 ト

- Language of propositional modal logic:
 - ► countable set of propositional variables P;
 - Two logical constants \top and \bot .
- Operators of propositional modal logic:
 - Boolean connectives: \rightarrow , \land ;

・ 同 ト ・ ヨ ト ・ ヨ ト

- Language of propositional modal logic:
 - ► countable set of propositional variables P;
 - Two logical constants \top and \bot .
- Operators of propositional modal logic:
 - Boolean connectives: \rightarrow , \wedge ;
 - ▶ Unary modal operator: □.

向下 イヨト イヨト

- Language of propositional modal logic:
 - ► countable set of propositional variables P;
 - Two logical constants \top and \bot .
- Operators of propositional modal logic:
 - Boolean connectives: \rightarrow , \wedge ;
 - ▶ Unary modal operator: □.
- All other Boolean connectives are defined as usual:

向下 イヨト イヨト

- Language of propositional modal logic:
 - ► countable set of propositional variables P;
 - Two logical constants \top and \bot .
- Operators of propositional modal logic:
 - Boolean connectives: \rightarrow , \wedge ;
 - ▶ Unary modal operator: □.
- ► All other Boolean connectives are defined as usual:

$$\blacktriangleright \ \neg \psi \quad := \quad \psi \to \bot;$$

向下 イヨト イヨト

- Language of propositional modal logic:
 - ► countable set of propositional variables P;
 - Two logical constants \top and \bot .
- Operators of propositional modal logic:
 - Boolean connectives: \rightarrow , \land ;
 - ▶ Unary modal operator: □.
- All other Boolean connectives are defined as usual:

$$\neg \psi := \psi \to \bot;$$

- 4 回 2 - 4 □ 2 - 4 □

- Language of propositional modal logic:
 - ► countable set of propositional variables P;
 - Two logical constants \top and \bot .
- Operators of propositional modal logic:
 - Boolean connectives: \rightarrow , \land ;
 - ▶ Unary modal operator: □.
- All other Boolean connectives are defined as usual:

$$\begin{array}{l} \bullet \ \neg\psi := \ \psi \to \bot; \\ \bullet \ \psi \lor \varphi := \ \neg (\neg \psi \land \neg \varphi); \end{array}$$

etc.

Syntax of modal logics Various modal logics

・ 同 ト ・ ヨ ト ・ ヨ ト

- Language of propositional modal logic:
 - ► countable set of propositional variables P;
 - Two logical constants \top and \bot .
- Operators of propositional modal logic:
 - Boolean connectives: \rightarrow , \land ;
 - ▶ Unary modal operator: □.
- All other Boolean connectives are defined as usual:

$$\neg \psi := \psi \to \bot;$$

etc.

• The dual modal operator \diamondsuit is defined as $\neg \Box \neg$

Syntax of modal logics Various modal logics

・ 同 ト ・ ヨ ト ・ ヨ ト

- Language of propositional modal logic:
 - ► countable set of propositional variables P;
 - Two logical constants \top and \bot .
- Operators of propositional modal logic:
 - Boolean connectives: \rightarrow , \land ;
 - ▶ Unary modal operator: □.
- All other Boolean connectives are defined as usual:

$$\neg \psi := \psi \to \bot;$$

$$\psi \lor \varphi := \neg (\neg \psi \land \neg \varphi);$$

etc.

- The dual modal operator \diamondsuit is defined as $\neg \Box \neg$
- \blacktriangleright \Box and \diamondsuit bind as \neg and the rest as usual

Syntax of modal logics Various modal logics

| 4 回 2 4 U = 2 4 U =

- Language of propositional modal logic:
 - ▶ countable set of propositional variables P;
 - Two logical constants \top and \bot .
- Operators of propositional modal logic:
 - Boolean connectives: \rightarrow , \land ;
 - ▶ Unary modal operator: □.
- All other Boolean connectives are defined as usual:

$$\neg \psi := \psi \to \bot;$$

- $\psi \lor \varphi := \neg (\neg \psi \land \neg \varphi);$
- etc.
- The dual modal operator \diamond is defined as $\neg \Box \neg$
- \Box and \diamondsuit bind as \neg and the rest as usual
- ► For us:

日本 (日本) (日本)

- Language of propositional modal logic:
 - ► countable set of propositional variables P;
 - Two logical constants \top and \bot .
- Operators of propositional modal logic:
 - Boolean connectives: \rightarrow , \land ;
 - ▶ Unary modal operator: □.
- All other Boolean connectives are defined as usual:

$$\neg \psi := \psi \to \bot;$$

etc.

- The dual modal operator \diamond is defined as $\neg \Box \neg$
- \blacktriangleright \Box and \diamondsuit bind as \neg and the rest as usual
- For us: \Box for provable and \diamond as consistent

Syntax of modal logics Various modal logics

・ロン ・聞と ・ほと ・ほと

æ

Various properties become naturally expressible

- Various properties become naturally expressible
- Formalized Modus Ponens

$$\Box(p
ightarrow q)
ightarrow (\Box p
ightarrow \Box q)$$

イロト イヨト イヨト イヨト

- Various properties become naturally expressible
- Formalized Modus Ponens

$$\Box(p
ightarrow q)
ightarrow (\Box p
ightarrow \Box q)$$

Uniform reflection

$$\Box p \rightarrow p$$

イロン イヨン イヨン イヨン

- Various properties become naturally expressible
- Formalized Modus Ponens

$$\Box(p
ightarrow q)
ightarrow (\Box p
ightarrow \Box q)$$

Uniform reflection

$$\Box p \rightarrow p$$

Gödel's second incompleteness theorem:

$$\Diamond \top \to \neg \Box \Diamond \top$$

Syntax of modal logics Various modal logics

< □ > < □ > < □ > < □ > < □ > .

æ

► The logic K

► The logic K

• All axioms of the form $\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$

・ロト ・回ト ・ヨト ・ヨト

► The logic K

- All axioms of the form $\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$
- All propositional tautologies as axioms

<ロ> (日) (日) (日) (日) (日)

The logic K

- All axioms of the form $\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$
- All propositional tautologies as axioms
- The only rules are Modus Ponens and Necessitation

イロト イヨト イヨト イヨト

The logic K

- All axioms of the form $\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$
- All propositional tautologies as axioms
- The only rules are Modus Ponens and Necessitation
- Non valid reasoning:

イロト イヨト イヨト イヨト

The logic K

- All axioms of the form $\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$
- All propositional tautologies as axioms
- The only rules are Modus Ponens and Necessitation
- Non valid reasoning:
 - ► Assume *p*

イロト イヨト イヨト イヨト

Syntax of modal logics Various modal logics

- 4 同 ト 4 臣 ト 4 臣 ト

æ

The logic K

- All axioms of the form $\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$
- All propositional tautologies as axioms
- The only rules are Modus Ponens and Necessitation
- Non valid reasoning:
 - Assume p
 - ▶ Derive □p by Necessitation

The logic K

- All axioms of the form $\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$
- All propositional tautologies as axioms
- The only rules are Modus Ponens and Necessitation
- Non valid reasoning:
 - ► Assume *p*
 - Derive $\Box p$ by Necessitation
 - Thus, conclude $p \rightarrow \Box p$

- 4 同 ト 4 臣 ト 4 臣 ト

Syntax of modal logics Various modal logics

- 4 回 2 - 4 □ 2 - 4 □

æ

The logic K

- All axioms of the form $\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$
- All propositional tautologies as axioms
- The only rules are Modus Ponens and Necessitation
- Non valid reasoning:
 - ► Assume *p*
 - ▶ Derive □*p* by Necessitation
 - Thus, conclude $p \rightarrow \Box p$
- Note: $\Box p \lor \neg \Box p$ is also an axiom

Syntax of modal logics Various modal logics

・ロン ・回 と ・ ヨン ・ モン

æ

$\blacktriangleright \mathbf{K} \vdash \Box A \land \Box B \iff \Box (A \land B)$

David Fernández Duque¹ and Joost J. Joosten² Provability as modality

Syntax of modal logics Various modal logics

▲口 → ▲圖 → ▲ 国 → ▲ 国 → □

æ

$\blacktriangleright \mathbf{K} \vdash \Box A \land \Box B \iff \Box (A \land B)$

▶ **Proof:** $A \rightarrow (B \rightarrow A \land B)$ is a tautology

David Fernández Duque¹ and Joost J. Joosten² Provability as modality

Syntax of modal logics Various modal logics

イロン イヨン イヨン イヨン

- $\blacktriangleright \mathbf{K} \vdash \Box A \land \Box B \leftrightarrow \Box (A \land B)$
- **Proof:** $A \rightarrow (B \rightarrow A \land B)$ is a tautology
- Necessitation and K axiom twice to obtain

Syntax of modal logics Various modal logics

イロン イヨン イヨン イヨン

- $\blacktriangleright \mathbf{K} \vdash \Box A \land \Box B \leftrightarrow \Box (A \land B)$
- **Proof:** $A \rightarrow (B \rightarrow A \land B)$ is a tautology
- Necessitation and K axiom twice to obtain

$$\Box A \to (\Box B \to \Box (A \land B))$$

Syntax of modal logics Various modal logics

- $\blacktriangleright \mathbf{K} \vdash \Box A \land \Box B \iff \Box (A \land B)$
- **Proof:** $A \rightarrow (B \rightarrow A \land B)$ is a tautology
- Necessitation and K axiom twice to obtain

$$\Box A \to (\Box B \to \Box (A \land B))$$

Use the tautology

 $(\Box A \to (\Box B \to \Box (A \land B))) \to (\Box A \land \Box B \to \Box (A \land B))$

・ロト ・回ト ・ヨト ・ヨト

2

Syntax of modal logics Various modal logics

- $\blacktriangleright \mathbf{K} \vdash \Box A \land \Box B \iff \Box (A \land B)$
- **Proof:** $A \rightarrow (B \rightarrow A \land B)$ is a tautology
- Necessitation and K axiom twice to obtain

$$\Box A \to (\Box B \to \Box (A \land B))$$

Use the tautology

 $(\Box A \to (\Box B \to \Box (A \land B))) \to (\Box A \land \Box B \to \Box (A \land B))$

・ロト ・回ト ・ヨト ・ヨト

• The other direction is similar starting with $A \wedge B \rightarrow A$

► The logic K4: as K but now adding all axioms of the form

$\Box A \to \Box \Box A$

▲口 → ▲圖 → ▲ 国 → ▲ 国 → □

► The logic K4: as K but now adding all axioms of the form

$\Box A \to \Box \Box A$

▶ The logic **GL**: as **K** but now adding all axioms of the form

$$\Box(\Box A \to A) \to \Box A$$

イロト イヨト イヨト イヨト

We shall see that

$\mathbf{K} + \{\Box(\Box A \to A) \to \Box A \mid A \text{ a modal formula}\} \vdash \Box B \to \Box \Box B.$

▲□→ ▲圖→ ▲厘→ ▲厘→

3

We shall see that

 $\mathbf{K} + \{\Box(\Box A \to A) \to \Box A \mid A \text{ a modal formula}\} \vdash \Box B \to \Box \Box B.$

Proof:

 $\mathbf{K}\vdash \Box B\rightarrow \Box (\Box (\Box B\wedge B)\rightarrow \Box B\wedge B)$

イロン イヨン イヨン イヨン

► We shall see that

 $\mathbf{K} + \{\Box(\Box A \to A) \to \Box A \mid A \text{ a modal formula}\} \vdash \Box B \to \Box \Box B.$

イロン イヨン イヨン イヨン

2

▶ Proof: $\mathbf{K} \vdash \Box B \rightarrow \Box (\Box (\Box B \land B) \rightarrow \Box B \land B)$

• Next, apply Löb to $\Box B \land B$.

We shall see that

 $\mathbf{K} + \{\Box (\Box A \to A) \to \Box A \mid A \text{ a modal formula}\} \vdash \Box B \to \Box \Box B.$

Proof:

$$\mathbf{K}\vdash \Box B \rightarrow \Box (\Box (\Box B \land B) \rightarrow \Box B \land B)$$

イロン イヨン イヨン イヨン

2

- Next, apply Löb to $\Box B \land B$.
- From now on we shall sometimes refer to GL as containing the axioms □A → □□A

We shall see that

 $\mathbf{K} + \{\Box (\Box A \to A) \to \Box A \mid A \text{ a modal formula}\} \vdash \Box B \to \Box \Box B.$

Proof:

$$\mathbf{K}\vdash \Box B \rightarrow \Box (\Box (\Box B \land B) \rightarrow \Box B \land B)$$

イロン イヨン イヨン イヨン

2

- Next, apply Löb to $\Box B \land B$.
- From now on we shall sometimes refer to GL as containing the axioms □A → □□A
- and sometimes as not containing those axioms

• We considered the liar λ :

◆□ > ◆□ > ◆臣 > ◆臣 > ○

• We considered the liar $\lambda: \lambda \leftrightarrow \neg Prv_{PA}(\lambda)$

- We considered the liar $\lambda: \lambda \leftrightarrow \neg Prv_{PA}(\lambda)$
- The liar is both true and independent

・ロ・ ・ 日・ ・ 日・ ・ 日・

- We considered the liar $\lambda: \lambda \leftrightarrow \neg \texttt{Prv}_{PA}(\lambda)$
- The liar is both true and independent
- What about the truth-teller?

- We considered the liar $\lambda: \lambda \leftrightarrow \neg Prv_{PA}(\lambda)$
- The liar is both true and independent
- What about the truth-teller?: $au \leftrightarrow \mathtt{Prv}_{\mathrm{PA}}(au)$

- We considered the liar $\lambda: \lambda \leftrightarrow \neg Prv_{PA}(\lambda)$
- The liar is both true and independent
- What about the truth-teller?: $au \leftrightarrow \mathtt{Prv}_{\mathrm{PA}}(au)$
- \blacktriangleright Now that we have a link to modal logic, we shall often write \Box_{PA} for \mathtt{Prv}_{PA}

- We considered the liar $\lambda: \lambda \leftrightarrow \neg Prv_{PA}(\lambda)$
- The liar is both true and independent
- What about the truth-teller?: $au \leftrightarrow \mathtt{Prv}_{\mathrm{PA}}(au)$
- \blacktriangleright Now that we have a link to modal logic, we shall often write \Box_{PA} for \mathtt{Prv}_{PA}

・ロト ・回ト ・ヨト ・ヨト

2

▶ By Löb we know $PA \vdash \Box_{PA}(\tau) \rightarrow \tau \implies PA \vdash \tau$

- We considered the liar $\lambda: \lambda \leftrightarrow \neg Prv_{PA}(\lambda)$
- The liar is both true and independent
- What about the truth-teller?: $au \leftrightarrow \mathtt{Prv}_{\mathrm{PA}}(au)$
- \blacktriangleright Now that we have a link to modal logic, we shall often write \Box_{PA} for \mathtt{Prv}_{PA}

- ▶ By Löb we know $PA \vdash \Box_{PA}(\tau) \rightarrow \tau \implies PA \vdash \tau$
- ► Thus, the truth-teller is both true and provable

- We considered the liar $\lambda: \lambda \leftrightarrow \neg Prv_{PA}(\lambda)$
- The liar is both true and independent
- What about the truth-teller?: $au \leftrightarrow \Pr_{PA}(au)$
- \blacktriangleright Now that we have a link to modal logic, we shall often write \Box_{PA} for \mathtt{Prv}_{PA}

イロン イヨン イヨン イヨン

- ▶ By Löb we know $PA \vdash \Box_{PA}(\tau) \rightarrow \tau \implies PA \vdash \tau$
- Thus, the truth-teller is both true and provable
- First proven by Löb

We shall give Löb's proof

David Fernández Duque¹ and Joost J. Joosten² Provability as modality

◆□ > ◆□ > ◆豆 > ◆豆 >

- We shall give Löb's proof
- for the sake of practicing with fixpoints and for beauty

イロト イヨト イヨト イヨト

- We shall give Löb's proof
- for the sake of practicing with fixpoints and for beauty
- based on a proof of the following theorem

|田・ (日) (日)

- We shall give Löb's proof
- for the sake of practicing with fixpoints and for beauty
- based on a proof of the following theorem
- Theorem Sinterklaas (Saint Nicholas) exists

▲圖 ▶ ▲ 臣 ▶ ▲ 臣 ▶

- We shall give Löb's proof
- for the sake of practicing with fixpoints and for beauty
- based on a proof of the following theorem
- Theorem Sinterklaas (Saint Nicholas) exists
- Proof:

(1日) (三) (三)

- We shall give Löb's proof
- for the sake of practicing with fixpoints and for beauty
- based on a proof of the following theorem
- Theorem Sinterklaas (Saint Nicholas) exists
- Proof:



David Fernández Duque¹ and Joost J. Joosten²

Provability as modality

Sinterklaas (Saint Nicholas) exists

・ロン ・四と ・ヨン ・ヨン

- Sinterklaas (Saint Nicholas) exists
- Proof: If this sentence is true, then Sinterklaas exists

イロト イヨト イヨト イヨト

- Sinterklaas (Saint Nicholas) exists
- Proof: If this sentence is true, then Sinterklaas exists

 $\blacktriangleright A \leftrightarrow (A \rightarrow S)$

- Sinterklaas (Saint Nicholas) exists
- Proof: If this sentence is true, then Sinterklaas exists
- $\blacktriangleright A \leftrightarrow (A \rightarrow S)$
- Suppose A (Assumption 1)

<ロ> (日) (日) (日) (日) (日)

- Sinterklaas (Saint Nicholas) exists
- Proof: If this sentence is true, then Sinterklaas exists
- $\blacktriangleright A \leftrightarrow (A \rightarrow S)$
- Suppose A (Assumption 1)
- Then $A \rightarrow S$ via Modus Ponens

- Sinterklaas (Saint Nicholas) exists
- Proof: If this sentence is true, then Sinterklaas exists
- $\blacktriangleright A \leftrightarrow (A \rightarrow S)$
- Suppose A (Assumption 1)
- Then $A \rightarrow S$ via Modus Ponens
- Using our assumption again, we get S

- 4 回 🕨 - 4 回 🕨 - 4 回 🕨

- Sinterklaas (Saint Nicholas) exists
- Proof: If this sentence is true, then Sinterklaas exists
- $\blacktriangleright A \leftrightarrow (A \rightarrow S)$
- Suppose A (Assumption 1)
- Then $A \rightarrow S$ via Modus Ponens
- Using our assumption again, we get S
- We conclude $A \rightarrow S$ discharging Assumption 1

- 4 回 🕨 - 4 回 🕨 - 4 回 🕨

- Sinterklaas (Saint Nicholas) exists
- Proof: If this sentence is true, then Sinterklaas exists
- $\blacktriangleright A \leftrightarrow (A \rightarrow S)$
- Suppose A (Assumption 1)
- Then $A \rightarrow S$ via Modus Ponens
- Using our assumption again, we get S
- We conclude $A \rightarrow S$ discharging Assumption 1
- ► This is just A

|田・ (日) (日)

- Sinterklaas (Saint Nicholas) exists
- Proof: If this sentence is true, then Sinterklaas exists
- $\blacktriangleright A \leftrightarrow (A \rightarrow S)$
- Suppose A (Assumption 1)
- Then $A \rightarrow S$ via Modus Ponens
- Using our assumption again, we get S
- We conclude $A \rightarrow S$ discharging Assumption 1
- This is just A
- Applying twice Modus Ponens we get S

(1日) (三) (三)

- Sinterklaas (Saint Nicholas) exists
- Proof: If this sentence is true, then Sinterklaas exists
- $\blacktriangleright A \leftrightarrow (A \rightarrow S)$
- Suppose A (Assumption 1)
- Then $A \rightarrow S$ via Modus Ponens
- Using our assumption again, we get S
- We conclude $A \rightarrow S$ discharging Assumption 1
- This is just A

Applying twice Modus Ponens we get S 1988



▶ Theorem (Löb) If $PA \vdash \Box_{PA} \psi \rightarrow \psi$, then $PA \vdash \psi$

▲□ → ▲圖 → ▲ 国 → ▲ 国 → →

æ.

- ▶ Theorem (Löb) If $PA \vdash \Box_{PA}\psi \rightarrow \psi$, then $PA \vdash \psi$
- ▶ **Proof** We consider χ with $PA \vdash \chi \leftrightarrow (\Box_{PA}\chi \rightarrow \psi)$ and reason in PA

イロン イヨン イヨン イヨン

- ▶ Theorem (Löb) If $PA \vdash \Box_{PA} \psi \rightarrow \psi$, then $PA \vdash \psi$
- ▶ **Proof** We consider χ with $PA \vdash \chi \leftrightarrow (\Box_{PA}\chi \rightarrow \psi)$ and reason in PA
- Thus, by necessitation and distribution

$$\Box_{\mathrm{PA}}\chi \rightarrow \left(\Box_{\mathrm{PA}}\Box_{\mathrm{PA}}\chi \rightarrow \Box\psi\right)$$

・ロン ・回と ・ヨン ・ヨン

- ▶ Theorem (Löb) If $PA \vdash \Box_{PA} \psi \rightarrow \psi$, then $PA \vdash \psi$
- ▶ **Proof** We consider χ with $PA \vdash \chi \leftrightarrow (\Box_{PA}\chi \rightarrow \psi)$ and reason in PA
- Thus, by necessitation and distribution

$$\Box_{\mathrm{PA}}\chi \rightarrow \left(\Box_{\mathrm{PA}}\Box_{\mathrm{PA}}\chi \rightarrow \Box\psi\right)$$

イロン イヨン イヨン イヨン

2

• By transitivity $\Box_{PA}\chi \rightarrow (\Box_{PA}\chi \rightarrow \Box\psi)$

- ▶ Theorem (Löb) If $PA \vdash \Box_{PA} \psi \rightarrow \psi$, then $PA \vdash \psi$
- ▶ **Proof** We consider χ with $PA \vdash \chi \leftrightarrow (\Box_{PA}\chi \rightarrow \psi)$ and reason in PA
- Thus, by necessitation and distribution

$$\Box_{\mathrm{PA}}\chi \rightarrow \left(\Box_{\mathrm{PA}}\Box_{\mathrm{PA}}\chi \rightarrow \Box\psi\right)$$

イロト イヨト イヨト イヨト

3

▶ By transitivity $\Box_{PA}\chi \rightarrow (\Box_{PA}\chi \rightarrow \Box\psi)$ which is just $\Box_{PA}\chi \rightarrow \Box\psi$

- ▶ Theorem (Löb) If $PA \vdash \Box_{PA} \psi \rightarrow \psi$, then $PA \vdash \psi$
- ▶ **Proof** We consider χ with $PA \vdash \chi \leftrightarrow (\Box_{PA}\chi \rightarrow \psi)$ and reason in PA
- Thus, by necessitation and distribution

$$\Box_{\mathrm{PA}}\chi \rightarrow \left(\Box_{\mathrm{PA}}\Box_{\mathrm{PA}}\chi \rightarrow \Box\psi\right)$$

- ▶ By transitivity $\Box_{PA}\chi \rightarrow (\Box_{PA}\chi \rightarrow \Box\psi)$ which is just $\Box_{PA}\chi \rightarrow \Box\psi$
- By assumption

$$\Box_{\rm PA}\chi \to \psi \tag{1}$$

イロト イヨト イヨト イヨト

3

- ▶ Theorem (Löb) If $PA \vdash \Box_{PA} \psi \rightarrow \psi$, then $PA \vdash \psi$
- ▶ **Proof** We consider χ with $PA \vdash \chi \leftrightarrow (\Box_{PA}\chi \rightarrow \psi)$ and reason in PA
- Thus, by necessitation and distribution

$$\Box_{\mathrm{PA}}\chi \rightarrow \left(\Box_{\mathrm{PA}}\Box_{\mathrm{PA}}\chi \rightarrow \Box\psi\right)$$

- ▶ By transitivity $\Box_{PA}\chi \rightarrow (\Box_{PA}\chi \rightarrow \Box\psi)$ which is just $\Box_{PA}\chi \rightarrow \Box\psi$
- By assumption

$$\Box_{\rm PA} \chi \to \psi \tag{1}$$

・ロト ・回ト ・ヨト ・ヨト

▶ Thus χ , whence by Nec. $\Box \chi$ and MP on (1) we get ψ

- ▶ Theorem (Löb) If $PA \vdash \Box_{PA} \psi \rightarrow \psi$, then $PA \vdash \psi$
- ▶ **Proof** We consider χ with $PA \vdash \chi \leftrightarrow (\Box_{PA}\chi \rightarrow \psi)$ and reason in PA
- Thus, by necessitation and distribution

$$\Box_{\mathrm{PA}}\chi \rightarrow \left(\Box_{\mathrm{PA}}\Box_{\mathrm{PA}}\chi \rightarrow \Box\psi\right)$$

- ▶ By transitivity $\Box_{PA}\chi \rightarrow (\Box_{PA}\chi \rightarrow \Box\psi)$ which is just $\Box_{PA}\chi \rightarrow \Box\psi$
- By assumption

$$\Box_{\rm PA}\chi \to \psi \tag{1}$$

・ 母 と ・ ヨ と ・ ヨ と

▶ Thus χ , whence by Nec. $\Box \chi$ and MP on (1) we get ψ

ſ

• Arithmetical realization: $f : \mathbb{P} \to \text{Sent}_{PA}$

æ.

- Arithmetical realization: $f : \mathbb{P} \to \text{Sent}_{PA}$
- ▶ We extend *f* to be defined on all modal formulas:

・ロト ・回 ト ・ヨト ・ヨトー

- Arithmetical realization: $f : \mathbb{P} \to \text{Sent}_{PA}$
- ▶ We extend *f* to be defined on all modal formulas:
 - f commutes with Boolean connectives;

・ロト ・回ト ・ヨト ・ヨト

- Arithmetical realization: $f : \mathbb{P} \to \text{Sent}_{PA}$
- ▶ We extend *f* to be defined on all modal formulas:
 - f commutes with Boolean connectives;
 - In particular, $f(\top) = \top$ and $f(\bot) = \bot$;

イロト イヨト イヨト イヨト

- Arithmetical realization: $f : \mathbb{P} \to \text{Sent}_{PA}$
- ▶ We extend *f* to be defined on all modal formulas:
 - f commutes with Boolean connectives;
 - In particular, $f(\top) = \top$ and $f(\bot) = \bot$;
 - $f(\Box A) = \Box_{\mathrm{PA}}f(A).$

イロン イヨン イヨン イヨン

- Arithmetical realization: $f : \mathbb{P} \to \text{Sent}_{PA}$
- We extend f to be defined on all modal formulas:
 - f commutes with Boolean connectives;
 - In particular, $f(\top) = \top$ and $f(\bot) = \bot$;
 - $f(\Box A) = \Box_{\mathrm{PA}}f(A).$
- ▶ **Theorem** If **GL** \vdash *A*, then for any arithmetical realization *f*, PA \vdash *f*(*A*)

- Arithmetical realization: $f : \mathbb{P} \to \text{Sent}_{PA}$
- We extend f to be defined on all modal formulas:
 - f commutes with Boolean connectives;
 - In particular, $f(\top) = \top$ and $f(\bot) = \bot$;
 - $f(\Box A) = \Box_{\mathrm{PA}}f(A).$
- ▶ **Theorem** If **GL** \vdash *A*, then for any arithmetical realization *f*, PA \vdash *f*(*A*)

・ 回 ト ・ ヨ ト ・ ヨ ト

Proof By induction on the proof A in GL

- Arithmetical realization: $f : \mathbb{P} \to \text{Sent}_{PA}$
- We extend f to be defined on all modal formulas:
 - f commutes with Boolean connectives;
 - In particular, $f(\top) = \top$ and $f(\bot) = \bot$;
 - $f(\Box A) = \Box_{\mathrm{PA}}f(A).$
- ▶ **Theorem** If **GL** \vdash *A*, then for any arithmetical realization *f*, PA \vdash *f*(*A*)

・ 回 ト ・ ヨ ト ・ ヨ ト

- Proof By induction on the proof A in GL
- Let Löb's rule be $(\Box A \rightarrow A)/A$

- Arithmetical realization: $f : \mathbb{P} \to \text{Sent}_{PA}$
- We extend f to be defined on all modal formulas:
 - f commutes with Boolean connectives;
 - In particular, $f(\top) = \top$ and $f(\bot) = \bot$;
 - $f(\Box A) = \Box_{\mathrm{PA}}f(A).$
- ▶ **Theorem** If **GL** \vdash *A*, then for any arithmetical realization *f*, PA \vdash *f*(*A*)
- Proof By induction on the proof A in GL
- Let Löb's rule be $(\Box A \rightarrow A)/A$
- It is easy to show that

$$\mathsf{K4} + \mathsf{LR} \vdash \Box (\Box A \to A) \to \Box A$$

・ 回 ト ・ ヨ ト ・ ヨ ト

- Arithmetical realization: $f : \mathbb{P} \to \text{Sent}_{PA}$
- We extend f to be defined on all modal formulas:
 - f commutes with Boolean connectives;
 - In particular, $f(\top) = \top$ and $f(\bot) = \bot$;
 - $f(\Box A) = \Box_{\mathrm{PA}}f(A).$
- ▶ **Theorem** If **GL** \vdash *A*, then for any arithmetical realization *f*, PA \vdash *f*(*A*)
- Proof By induction on the proof A in GL
- Let Löb's rule be $(\Box A \rightarrow A)/A$
- It is easy to show that

$$\mathsf{K4} + \mathsf{LR} \vdash \Box (\Box A \to A) \to \Box A$$



<ロ> <同> <同> < 同> < 同> < 同><<