Provability Logics and Applications
Day 1
Provability as modality

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ESSLLI Tutorial, Opole

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- By provable Σ_1 completeness: $\mathtt{Prv}_{\mathrm{PA}}(\mathtt{Prv}_{\mathrm{PA}}(\lambda))$, that is $\mathtt{Prv}_{\mathrm{PA}}(\neg\lambda)$

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By deduction (and the formalized version)

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- **Theorem** If $\mathrm{PA} \vdash \mathtt{Prv}_{\mathrm{PA}}(\varphi) \to \varphi$, then $\mathrm{PA} \vdash \varphi$
- PA is as modest about it's own correctness as it could possibly be
- **Proof:** By contraposition supposing $\mathrm{PA} \nvdash \varphi$
- Thus $\mathrm{PA} + \neg\varphi$ is consistent
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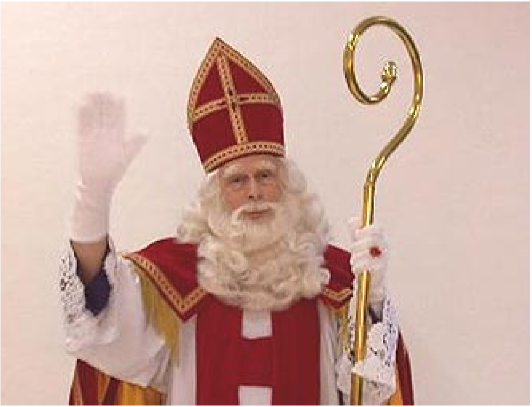

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- We extend f to be defined on all modal formulas:
  - f commutes with Boolean connectives;
  - In particular, $f(\top) = \top$ and $f(\bot) = \bot$;
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- **Theorem** If **GL** $\vdash A$, then for any arithmetical realization f, $\mathrm{PA} \vdash f(A)$
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