# Provability Logics and Applications Day 2 Completeness results for **GL**

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Soundness Completeness

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- ► actually, this holds for any theory extending, say I∆<sub>0</sub> + exp, which is also called EA, or Kalmar Elementary Arithmetic
- extending in a very strong sense (like ZFC, etc)
- Thus, arithmetical soundness of GL is very stable

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### Do we also have arithmetical completeness?

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- Provability in PA is undecidable:
- GL is decidable
- How do we get so many (different) PA unprovable statements?
- We get this via another kind of semantics for **GL**.

Kripke semantics Soundness Completeness

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## ► **GL** allows possible world semantics

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Kripke semantics Soundness Completeness

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  - Non-confluent:  $x \uparrow$  is a linear order
  - Where  $x \uparrow := \{y \in W \mid y \succ x\}$
  - Well-founded: impossible to have an infinite descending chain

$$x_0 \succ x_1 \succ x_2 \succ \ldots$$

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Note that our trees are like hang-trees

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Why do we use this unconventional Kripke frames representation?

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- Why do we use this unconventional Kripke frames representation?
- To link to ordinals

We do allow infinite increasing chains: x<sub>0</sub> ≺ x<sub>1</sub> ≺ x<sub>2</sub>... in GL frames

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- "Shortest infinite increasing chain":  $\omega$

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- ► Transfinite induction:  $\forall \alpha \in On \ (\forall \beta < \alpha \ \Phi(\beta) \rightarrow \Phi(\alpha)) \longrightarrow \forall \alpha \in On \ \Phi(\alpha)$

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Kripke semantics Soundness Completeness

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  - $W, w \Vdash \Box A \text{ iff } (\forall v \ [w \succ v \rightarrow v \Vdash A])$

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- We also write  $W \models A$  for short
- Three notions of truth!

We observe:

#### $F \models Ax(GL)$ & $GL \vdash A \implies F \models A$

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We observe:

## $F \models Ax(GL)$ & $GL \vdash A \implies F \models A$

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Proof:

We observe:

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 Proof: frame validity is closed under Modus Ponens and Necessitation We observe:

## $F \models Ax(GL)$ & $GL \vdash A \implies F \models A$

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- Proof: frame validity is closed under Modus Ponens and Necessitation
- Application:  $\mathbf{GL} \nvDash \Box p \rightarrow p$

Kripke semantics Soundness Completeness

We observe:

# $F \models Ax(GL)$ & $GL \vdash A \implies F \models A$

- Proof: frame validity is closed under Modus Ponens and Necessitation
- Application:  $\mathbf{GL} \nvDash \Box p \rightarrow p$
- We shall formulate sufficient (and necessary) conditions for F \= Ax(GL)

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The axioms □(A → B) → (□A → □B) hold on any Kripke frame, in particular, on tree-like, well-founded trees

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- The axioms □(A → B) → (□A → □B) hold on any Kripke frame, in particular, on tree-like, well-founded trees
- The axiom  $\Box A \rightarrow \Box \Box A$  holds on all transitive frames

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- The axioms □(A → B) → (□A → □B) hold on any Kripke frame, in particular, on tree-like, well-founded trees
- The axiom  $\Box A \rightarrow \Box \Box A$  holds on all transitive frames
- ► Also, if a frame validates □A → □□A, then it must be transitive

• If  $\langle W, \succ \rangle$  is well-founded and transitive, then

 $\langle W, \succ 
angle \models \mathsf{Ax}(\mathsf{GL})$ 

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• If  $\langle W, \succ \rangle$  is well-founded and transitive, then

$$\langle W, \succ \rangle \models \mathsf{Ax}(\mathsf{GL})$$

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Proof: we only need to check Löb's axiom.

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$$\operatorname{ord}(w) := \sup \{ \operatorname{ord}(v) + 1 \mid w \succ v \}$$

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By transfinite induction on ord(w) we prove:

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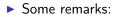
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• sup 
$$\emptyset = 0$$

- By transfinite induction on ord(w) we prove:
- $\langle W, \succ, V \rangle, w \Vdash \Box (\Box A \to A) \to \Box A$  for any w

Kripke semantics Soundness Completeness

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Kripke semantics Soundness Completeness

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#### Some remarks:

Dealing with well-foundedness within a decidable logic

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- Some remarks:
- Dealing with well-foundedness within a decidable logic
- We can avoid the use of transfinite induction

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#### Some remarks:

- Dealing with well-foundedness within a decidable logic
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- Some remarks:
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### Some remarks:

- Dealing with well-foundedness within a decidable logic
- We can avoid the use of transfinite induction
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### Some remarks:

- Dealing with well-foundedness within a decidable logic
- We can avoid the use of transfinite induction
- Recurrent theme:

$$\Box(\Box A \to A) \to \Box A$$

versus transfinite induction

Kripke semantics Soundness Completeness

#### For

# $F \models \mathsf{Ax}(\mathsf{GL})$

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## it is sufficient that $\succ$ is transitive and well-founded

David Fernández Duque<sup>1</sup> and Joost J. Joosten<sup>2</sup> Completeness results for GL

Kripke semantics Soundness Completeness

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#### For

# $F \models \mathsf{Ax}(\mathsf{GL})$

it is sufficient that ≻ is transitive and well-founded
≻ is transitive and well-founded on F

Kripke semantics Soundness Completeness

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#### For

# $F \models \mathsf{Ax}(\mathsf{GL})$

it is sufficient that  $\succ$  is transitive and well-founded

- $\blacktriangleright$  > is transitive and well-founded on F
- ► We shall now see: also necessary

► Theorem If (W, ≻) ⊨ Löb then ≻ is transitive and well-founded.

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- ► Theorem If (W, ≻) ⊨ Löb then ≻ is transitive and well-founded.
- **Proof:** We observed:

$$F \models \mathsf{Ax}(\mathsf{GL}) \And \mathsf{GL} \vdash A \implies F \models A$$

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Likewise, for any other modal logic L

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- Löb proves  $\Box A \rightarrow \Box \Box A$
- ▶ So, if  $\langle W, \succ \rangle \models$  Löb, then  $\succ$  is transitive

# ► Theorem If (W, ≻) ⊨ Löb then ≻ is transitive and well-founded.

David Fernández Duque<sup>1</sup> and Joost J. Joosten<sup>2</sup> Completeness results for GL

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- ► Theorem If (W, ≻) ⊨ Löb then ≻ is transitive and well-founded.
- Suppose  $\succ$  is transitive but ill-founded

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- ► Theorem If (W, ≻) ⊨ Löb then ≻ is transitive and well-founded.
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- $\blacktriangleright$  We'll exhibit V so that an instance of Löb does not hold on  $\langle W, \succ, V \rangle$

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• Consider  $x_0 \succ x_1 \succ x_2 \succ x_3 \succ \ldots$ 

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- Consider  $x_0 \succ x_1 \succ x_2 \succ x_3 \succ \ldots$
- Define

$$y \in V(p)$$
 :  $\iff y \notin \{x_0, x_1, x_2, \ldots\}$ 

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Now

$$\langle W, \succ, V \rangle, x_0 \nvDash \Box (\Box p \to p) \to \Box p$$

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Kripke semantics Soundness Completeness

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Soundness

# $\mathbf{GL} \vdash A \implies F \models A$

for any **GL** frame F

Kripke semantics Soundness Completeness

Soundness

$$\mathsf{GL} \vdash A \implies F \models A$$

for any  ${\bf GL}$  frame  ${\it F}$ 

Completeness:

$$\forall^{\mathsf{GL} \text{ frame}} F F \models A \implies \mathsf{GL} \vdash A$$

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also holds (Segerberg [1971])

Soundness

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Kripke semantics Soundness Completeness

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- A root is x so that  $\forall y \ (y \neq x \rightarrow x \succ y)$

Kripke semantics Soundness Completeness

Soundness

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also holds (Segerberg [1971])

- Actually we have completeness w.r.t. rooted finite trees
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- Application:

 $\mathbf{GL} \vdash \Box A \quad \Rightarrow \quad \mathbf{GL} \vdash A$ 

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Embedding Kripke models Embedding a linear frame On the Solovay function

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# ▶ Completeness: $\forall f \text{ PA} \vdash f(A) \implies \mathbf{GL} \vdash A$

- ▶ Completeness:  $\forall f \text{ PA} \vdash f(A) \implies \mathbf{GL} \vdash A$
- ▶ We shall prove: **GL**  $\nvdash$   $A \rightarrow \exists f \text{ PA} \nvdash f(A)$

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Arithmetical semantics	Embedding Kripke models
Kripke semantics for GL	Embedding a linear frame
On arithmetical completeness	On the Solovay function

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- We shall prove: **GL**  $\nvdash A \rightarrow \exists f \text{ PA} \nvdash f(A)$
- How to get such sentences f(A)
- It took us already quite some effort to obtain one such sentence Con(PA)!

Embedding Kripke models Embedding a linear frame On the Solovay function

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## We will embed Kripke models into arithmetic

David Fernández Duque<sup>1</sup> and Joost J. Joosten<sup>2</sup> Completeness results for GL

Arithmetical semantics	Embedding Kripke models
Kripke semantics for GL	Embedding a linear frame
On arithmetical completeness	On the Solovay function

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- ▶ We will embed Kripke models into arithmetic
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Arithmetical semantics	Embedding Kripke models
Kripke semantics for GL	Embedding a linear frame
On arithmetical completeness	On the Solovay function

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• enumerate 
$$W := \{1, 2, \ldots, n\}$$

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Arithmetical semantics	Embedding Kripke models
Kripke semantics for GL	Embedding a linear frame
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- ▶ we assign to each i ∈ W an arithmetical sentence λ<sub>i</sub> that "posseses the same algebraic properties as the worlds i ∈ W"

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• PA 
$$\vdash \lambda_i \rightarrow \neg \lambda_j$$
 for  $i \neq j$ ;

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Arithmetical semantics	Embedding Kripke models
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Arithmetical semantics	Embedding Kripke models
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▶ By setting  $f(p) := \bigvee_{i \Vdash p} \lambda_i$  we can prove **Truth Lemma** 

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Arithmetical semantics	Embedding Kripke models
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$$i \Vdash \psi \implies \mathrm{PA} \vdash \lambda_i \to f(\psi)$$

Arithmetical semantics	Embedding Kripke models
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• 
$$i \Vdash \psi \implies \mathrm{PA} \vdash \lambda_i \to f(\psi)$$

• **Proof:** By induction on  $\psi$  after induction building:

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$$i \Vdash \neg \psi \implies \mathrm{PA} \vdash \lambda_i \to \neg f(\psi)$$

Arithmetical semantics	Embedding Kripke models
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On arithmetical completeness	On the Solovay function

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$$\bullet \ i \Vdash \psi \implies \mathrm{PA} \vdash \lambda_i \to f(\psi)$$

**Proof:** By induction on  $\psi$  after induction building:

 $i \Vdash \neg \psi \implies \mathrm{PA} \vdash \lambda_i \to \neg f(\psi)$ 

If we can see (outside PA of course) that N ⊨ Con<sub>PA</sub>(λ<sub>1</sub>) we would be done

## ▶ We can embed linear frames easily (ESSLLI 2003)

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Arithmetical semantics	Embedding Kripke models
Kripke semantics for GL	Embedding a linear frame
On arithmetical completeness	On the Solovay function

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- Suppose we wish to embed  $1 \succ 2 \succ 3 \succ \ldots \succ n$
- ▶ We can map i to  $\mathsf{Bew}_{\mathrm{PA}}^{n+1-i}(\ulcorner 0 = 1\urcorner) \land \mathsf{Con}_{\mathrm{PA}}^{n-i}(\ulcorner 1 = 1\urcorner)$

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▶ Similarly,  $\mathsf{Con}_{\mathrm{PA}}(\lambda_1) \leftrightarrow \mathsf{Con}_{\mathrm{PA}}^{n+1}(\ulcorner1=1\urcorner)$ 

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- using arithmetical soundness of GL we can prove the three properties
- ▶ Similarly,  $\mathsf{Con}_{\mathrm{PA}}(\lambda_1) \leftrightarrow \mathsf{Con}_{\mathrm{PA}}^{n+1}(\ulcorner 1 = 1\urcorner)$
- Thus (outside PA)

$$\mathbb{N}\models\mathsf{Con}_{\mathrm{PA}}(\lambda_1)$$

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Arithmetical semantics	Embedding Kripke models
Kripke semantics for GL	Embedding a linear frame
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▶ If the frame is not linear, an additional trick is needed for  $\mathbb{N} \models \mathsf{Con}_{\mathrm{PA}}(\lambda_1)$ 

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- If the frame is not linear, an additional trick is needed for  $\mathbb{N} \models \mathsf{Con}_{\mathrm{PA}}(\lambda_1)$
- Solovay (1976) achieves this in his beautiful proof by adding an additional root 0 on top of the rest of the model.

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• The  $\lambda_i$  will be constructed in a self-referential fashion

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- The  $\lambda_i$  will be constructed in a self-referential fashion
- Very much like the current European refugee regulations