Provability Logics and Applications
Day 2
Completeness results for **GL**

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ESSLLI Tutorial, Opole

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- Thus, arithmetical soundness of **GL** is very stable

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- We get this via another kind of semantics for **GL**.

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  - Well-founded: impossible to have an infinite descending chain

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- To link to ordinals

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- Transfinite induction:
  $\forall \alpha \in \mathrm{On}\ (\forall \beta < \alpha\ \Phi(\beta) \rightarrow \Phi(\alpha)) \ \longrightarrow \ \forall \alpha \in \mathrm{On}\ \Phi(\alpha)$

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  - $W, w \Vdash \Box A$ iff $(\forall v\ [w \succ v \to v \Vdash A])$

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- Three notions of truth!

- We observe:

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- We shall formulate sufficient (and necessary) conditions for $F \models \mathrm{Ax}(\mathbf{GL})$

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- The axioms $\Box(A \to B) \to (\Box A \to \Box B)$ hold on any Kripke frame, in particular, on tree-like, well-founded trees
- The axiom $\Box A \to \Box\Box A$ holds on all transitive frames
- Also, if a frame validates $\Box A \to \Box\Box A$, then it must be transitive

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- $\langle W, \succ, V \rangle, w \Vdash \Box(\Box A \to A) \to \Box A$ for any w

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- We shall now see: also necessary

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- Now
$$\langle W, \succ, V \rangle, x_0 \not\models \Box(\Box p \to p) \to \Box p$$

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- Application:

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- How to get such sentences $f(A)$
- It took us already quite some effort to obtain one such sentence $\mathrm{Con}(\mathrm{PA})$!

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- If we can see (outside PA of course) that $\mathbb{N} \models \mathsf{Con}_{\mathrm{PA}}(\lambda_1)$ we would be done

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- Similarly, $\mathsf{Con}_{\mathrm{PA}}(\lambda_1) \leftrightarrow \mathsf{Con}_{\mathrm{PA}}^{n+1}(\ulcorner 1 = 1 \urcorner)$
- Thus (outside PA)

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- Very much like the current European refugee regulations