Provability Logics and Applications
Day 3
Polymodal logics

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ESSLLI Tutorial, Opole

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- Calling the first infinite ordinal ω, let's see some pictures of ω, $\omega + \omega = \omega \cdot 2$, etc   In particular: $1 + \omega \neq \omega + 1$

- **Theorem [Cantor]**: Each ordinal α can be *uniquely* written as

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- Note that 0 is denoted by the empty sum!

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- **Proof**: By induction on the length of the GLP_Λ proof of φ

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- **Corrolaries**: decidability, PSPACE completeness, interpolation, fixpoint theorem, etc.

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- Turing progressions can be used for an ordinal analysis:
- "how often should I iterate a finitistic base theory T as to approximate a target theory U: $T_\xi \approx U$"

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- **Proposition:** $T + \langle n+1 \rangle_T \top$ is a Π_{n+1} conservative extension of $T + \{\langle n \rangle_T^k \top \mid k \in \omega\}$.

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Definition (Worms, Worm, Worm_α)

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Similarly, we inductively define for each ordinal α the set of worms Worm_α where all ordinals are at least α as $\top \in \mathsf{Worm}_\alpha$ and $A \in \mathsf{Worm}_\alpha \wedge \beta \geq \alpha \Rightarrow \langle\beta\rangle A \in \mathsf{Worm}_\alpha$.

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- Worms owe their name to the heroic *worm battle*

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- **Proposition** For each ordinal $\alpha < \varepsilon_0$ there is some GLP_ω-worm A such that $T + A$ is Π_1 equivalent to T_α.
- To get generalizations of this lemma beyond ε_0 one should consider more than ω modalities.

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3. *If* $A \in \mathrm{Worm}_{\alpha+1}$*, then* GLP $\vdash A \wedge \langle\alpha\rangle B \leftrightarrow A\alpha B$;

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- We will order the worms based on these sort of implications