Provability Logics and Applications Day 3 Polymodal logics

David Fernández Duque¹ and Joost J. Joosten²

1: Universidad de Sevilla;

2: Universitat de Barcelona

Monday 13-08-2012 ESSLLI Tutorial, Opole

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A very brief word on ordinals The logics ${\rm GLP}_\Lambda$ A reduction to ${\rm GLP}_\omega$

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- ► Two order-types (W₁, ≺₁) and (W₂, ≺₂) are (order) isomorphic whenever there is a bijection f : W₁ → W₂ with

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Ordinals can be seen as equivalence classes of well-orders under order isomorphisms

• Calling the first infinite ordinal ω , let's see some pictures of ω , $\omega + \omega = \omega \cdot 2$. etc ・ロン ・回 と ・ ヨ と ・ ヨ と

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 Ordinals can be seen as equivalence classes of well-orders under order isomorphisms

► Calling the first infinite ordinal ω , let's see some pictures of ω , $\omega + \omega = \omega \cdot 2$, etc In particular: $1 + \omega \neq \omega + 1$

3

Theorem [Cantor]: Each ordinal α can be uniquely written as

$$\alpha := \omega^{\alpha_1} + \dots \omega^{\alpha_n}$$

where $\alpha_1 \geq \ldots \geq \alpha_n$.

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Note that 0 is denoted by the empty sum!

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$$[\alpha](\varphi \to \psi) \to ([\alpha]\varphi \to [\alpha]\psi) \qquad (\alpha < \Lambda)$$

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Rules: Modus ponens and Necessitation for all modalities

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 Intended reading of [α]:

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- Rules: Modus ponens and Necessitation for all modalities
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A very brief word on ordinal: The logics GLP_A A reduction to GLP_ω

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- **Proof**: By induction on the length of the GLP_A proof of φ

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- Corrolaries: decidability, PSPACE completeness, interpolation, fixpoint theorem, etc.

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▶ Gödel 2: for sound recursive theories *T* that can code syntax:

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Turing progressions:

$$\begin{array}{rcl} T_0 & := & T; \\ T_{\alpha+1} & := & T_{\alpha} + \operatorname{Con}(T_{\alpha}); \end{array}$$

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Turing progressions:

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- Turing progressions can be used for an ordinal analysis:
- ► "how often should I iterate a finitistic base theory T as to approximate a target theory U: $T_{\mathcal{E}} \approx U$ "

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- For $n \in \mathbb{N}$ we see $T_n \equiv T + \diamondsuit_T^n \top$
- Transfinite progressions are not expressible in the modal language with just one modal operator.
- ► However:
- ▶ **Proposition:** $T + \langle n+1 \rangle_T \top$ is a \prod_{n+1} conservative extension of $T + \{\langle n \rangle_T^k \top \mid k \in \omega\}$.

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For Ordinal analyses and Turing progressions a particular interest lies in GLP^0_Λ : the closed fragment

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Definition (Worms, Worm, Worm_{α})

By Worm we denote the set of *worms* of GLP which is inductively defined as $\top \in$ Worm and $A \in$ Worm $\Rightarrow \langle \alpha \rangle A \in$ Worm.

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Similarly, we inductively define for each ordinal α the set of worms $\operatorname{Worm}_{\alpha}$ where all ordinals are at least α as $\top \in \operatorname{Worm}_{\alpha}$ and $A \in \operatorname{Worm}_{\alpha} \land \beta \ge \alpha \Rightarrow \langle \beta \rangle A \in \operatorname{Worm}_{\alpha}$.

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- ▶ Worms form the backbone of GLP⁰
- Each closed formula is provably equivalent to a Boolean combination of worms
- GLP^0_{Λ} is decidable if Λ is
- Decision procedure factors through the worms
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- The computation of a proof-theoretic ordinal is largely done via worms
- ► Worms owe their name to the heroic *worm battle*

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- To get generalizations of this lemma beyond ε₀ one should consider more than ω modalities.

Turing progressions
Mighty worms
Worms and Turing progressions
Basic worm manipulations

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Sloppy notations for worms:

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- $\langle \omega \rangle 0 \omega$,
- or $\langle \omega \rangle \langle 0 \rangle \langle \omega \rangle \top$.

Transfinite provability logics Mighty worms Worms Worms Basic worm manipulations

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Lemma

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- 2. For worms A and B, if $\beta < \alpha$, then GLP $\vdash (\langle \alpha \rangle A \land \langle \beta \rangle B) \leftrightarrow \langle \alpha \rangle (A \land \langle \beta \rangle B);$

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- 2. For worms A and B, if $\beta < \alpha$, then GLP $\vdash (\langle \alpha \rangle A \land \langle \beta \rangle B) \leftrightarrow \langle \alpha \rangle (A \land \langle \beta \rangle B);$
- 3. If $A \in Worm_{\alpha+1}$, then $GLP \vdash A \land \langle \alpha \rangle B \leftrightarrow A \alpha B$;

▶ 'The axiom $\langle \alpha \rangle \psi \rightarrow [\beta] \langle \alpha \rangle \psi$ for $\alpha < \beta$, implies the existence of many smaller worms':

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- We will order the worms based on these sort of implications

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