Provability Logics and Applications
Day 4
The closed fragment

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3. *If* $A \in \mathrm{Worm}_{\alpha+1}$, *then* $\mathrm{GLP} \vdash A \wedge \langle\alpha\rangle B \leftrightarrow A\alpha B$*;*

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- We will order the worms based on these sort of implications

Definition ($<, <_\alpha$)

We define a relation $<_\alpha$ on $\mathrm{Worm}_\alpha \times \mathrm{Worm}_\alpha$ by

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Instead of $<_0$ we shall often write $<$.

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Proof.

This is immediate if we conceive α^n as $\lambda\alpha\lambda \ldots \lambda\alpha\lambda$. □

- **Lemma**: Let $A = A_k\alpha \ldots A_1\alpha A'$ be in BNF with $\alpha = \min(A)$, and each $A_i \in \mathsf{Worm}_{\alpha+1}$. Moreover, let B be in BNF. We have that

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- **Proof**: By induction on k.

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$$A <_\alpha B \iff A <_0 B$$

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Proof.

Apply (inductively) the lemma to ‘subworms in violating order’ □

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- If $B_n \geq_0 B_{n+1} \geq_0 \ldots$ has an infinite descending subsequence $B_n >_0 B_{n_1} >_0 B_{n_2} >_0 \ldots$ this contradicts minimality via

$$C_0 >_0 C_1 >_0 \ldots >_0 B_n >_0 B_{n_1} >_0 B_{n_2} >_0 \ldots$$

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- write each C_i as $A_i 0 B_i$ with $B_i \in \text{Worm}_1$, and the $A_i 0$ part possibly empty
- Clearly, we have $B_n \geq_0 B_{n+1} \geq_0 \ldots$
- If $B_n \geq_0 B_{n+1} \geq_0 \ldots$ has an infinite descending subsequence $B_n >_0 B_{n_1} >_0 B_{n_2} >_0 \ldots$ this contradicts minimality via

$$C_0 >_0 C_1 >_0 \ldots >_0 B_n >_0 B_{n_1} >_0 B_{n_2} >_0 \ldots$$

- If the chain remains stable at some $k+1$, then

$$C_0 >_0 \ldots >_0 C_k >_0 A_{k+1} >_0 A_{k+2} >_0 \ldots$$

also contradicting the minimality assumption

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- Before we can expose models for the closed fragment, we need a bit more facts about ordinals

- On ordering $\langle W, \prec \rangle$ is a well-order if
  - $\prec$ is transitive: $x \prec y$ and $y \prec z$ implies $x \prec z$
  - $\prec$ is linear: $x = y$ or $x \prec y$ or $y \prec x$
  - $\prec$ is well-founded on X: no infinite descending chain $\ldots x_3 \prec x_2 \prec x_1 \prec x_0$ within X
    under the axiom of choice equivalent to "each non-empty subset of X has a $\prec$-minimal element"
- Two order-types $\langle W_1, \prec_1 \rangle$ and $\langle W_2, \prec_2 \rangle$ are (order) isomorphic whenever there is a bijection $f : W_1 \to W_2$ with

$$x \prec_1 y \quad \Leftrightarrow \quad f(x) \prec_2 f(y).$$

- Ordinals can be seen as equivalence classes of well-orders under order isomorphisms

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- With these kind of definitions we can now prove statements like $1 + \omega \neq \omega + 1$

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- Many useful properties can now be proven and all coincide with the pictures we drew

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$$\alpha := \omega^{\alpha_1} + \ldots \omega^{\omega_n}$$

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- Clearly, the CNF of ε_0 is not too informative as

$$\varepsilon_0 = \omega^{\varepsilon_0}$$