Provability Logics and Applications Day 5 Ordinal analysis

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Turing progressions

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- Proposition For each ordinal α < ε₀ there is some GLP_ω-worm A such that T + A is Π₁ equivalent to T_α.

Turing progressions

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► Recall:

- Proposition For each ordinal α < ε₀ there is some GLP_ω-worm A such that T + A is Π₁ equivalent to T_α.
- In this lecture: compute order-types of worms

▶ **Definition**: $\mathsf{RFN}_{\Pi_n}(T) := \{\Box_T \pi(\dot{x}) \to \pi(x) \mid \pi \in \Pi_n\}$

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- **Proposition** : $RFN_{\Pi_n}(T)$ can be written as a single formula

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• Theorem : EA $\vdash \langle n \rangle_T \top \equiv \mathsf{RFN}_{\Pi_{n+1}}(T)$

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• Theorem : $EA \vdash \langle n \rangle_T \top \equiv \mathsf{RFN}_{\Pi_{n+1}}(T)$

▶ **Proof** : Reason in EA and suppose both $\langle n \rangle_T \top$ and $\Box_T \pi$.

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- Whence $[0]_T \neg \pi$ thus $\text{True}_{\Pi_{n+1}}(\neg \pi)$ by reflection
- The latter contradicts the assumption that $True_{\Pi_n}(\pi)$

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- ► By standard techniques in proof-theory this can be lowered to Σ_n induction

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- Thus, $EA^+ \vdash \langle n+1 \rangle_{EA} \top \equiv I \Sigma_n$
- See how expressible the closed fragment is!

Fundamental sequences The reduction property and its formalization

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We already saw that

$\mathbf{EA} + \langle \mathbf{1} \rangle_{\mathbf{EA}} \top \equiv_{\Pi_1} \mathbf{EA} + \{ \langle \mathbf{0} \rangle^n \top \mid n \in \omega \}$

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Fundamental sequences The reduction property and its formalization

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Fundamental sequences The reduction property and its formalization

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Fundamental sequences The reduction property and its formalization

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- It not hard to see that:
- For any worm A and numbers k and n we have $Q_n^k(A)$ is again a worm
- Moreover:

$$\mathsf{GLP} \vdash \langle n+1 \rangle A \to \langle 0 \rangle Q_n^k(A)$$

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That is

 $Q_n^k(A) <_0 \langle n+1 \rangle A$

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▶ Moreover, provable in EA⁺!

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- Thus both theories are equi-consistent:

 $\mathrm{EA}^{+} \vdash \langle 0 \rangle_{\mathrm{EA}} \langle n+1 \rangle_{\mathrm{EA}} A \iff \forall k \ \langle 0 \rangle_{\mathrm{EA}} Q_{n}^{k}(A)$

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This is called the formalized reduction property

 $\begin{array}{l} \textbf{Proof-theoretical ordinals} \\ \textbf{A consistency proof for } \mathrm{PA} \end{array}$

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$$|\mathbf{PA} + \mathsf{Con}(\mathbf{PA})|_{\Pi_1^0} = \varepsilon_0 \cdot 2$$

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- However, Π_1^0 are much more fine-grained
- ► $|PA + Con(PA)|_{\Pi_1^0} = \varepsilon_0 \cdot 2$
- ▶ whereas |PA|_{Π⁰₁} = ε₀

Proof-theoretical ordinals A consistency proof for PA

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$$\forall y \ \left(\forall \ y' {\prec} y \ \varphi(\vec{x}, y') \rightarrow \varphi(\vec{x}, y)\right) \ \rightarrow \ \forall y \ \varphi(\vec{x}, y)$$

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$$\forall y \ \bigl(\forall \, y' {\prec} y \ \varphi(\vec{x}, y') \to \varphi(\vec{x}, y)\bigr) \ \to \ \forall y \ \varphi(\vec{x}, y)$$

• with $\varphi(\vec{x}, y) \in X$

Proof-theoretical ordinals A consistency proof for PA

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▶ **Theorem**: $EA^+ + TI[\Pi_1^0, \varepsilon_0] \vdash Con(PA)$

Proof-theoretical ordinals A consistency proof for PA

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- ▶ **Theorem**: $EA^+ + TI[\Pi_1^0, \varepsilon_0] \vdash Con(PA)$
- ▶ Proof: We reason in EA⁺

Proof-theoretical ordinals A consistency proof for PA

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Our instantiation:

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Thus, we set out to prove

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David Fernández Duque¹ and Joost J. Joosten² Ordinal analysis

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