Provability logics and applications

Day 1: Provability as modality

1. Give formal proofs to the extend that $\mathbf{K} \vdash \Box A \wedge \Box B \leftrightarrow \Box(A \wedge B)$. (Hints are in the slides.)

2. Let Löb's rule –we write LR– be $\Box A \to A/A$.

   (a) Show that $\mathbf{K} + \mathsf{LR} = \mathbf{K}$

   (b) Show that $\mathbf{K4} + \mathsf{LR} = \mathbf{GL}$

3. Show that $\mathbf{GL} \vdash \Box A \to \Box\Box A$. (Hints are in the slides.)

4. Let λ be Gödel's liar sentence so that $\mathrm{PA} \vdash \neg\mathtt{Prv}_{\mathrm{PA}}(\lambda) \leftrightarrow \lambda$

   (a) Show that $\mathrm{PA} \vdash \lambda \leftrightarrow \mathtt{Con}_{\mathrm{PA}}$.

   (b) Show that if PA is consistent, then $\mathrm{PA} \nvdash \lambda$.

5. **Löb's Theorem**
   In the lecture we have proven Löb's Theorem by considering a fixpoint of

$$\mathsf{Bew}_{\mathsf{PA}}(x) \to A.$$

   (a) Formulate Gödel's Second Incompleteness Theorem.

   (b) Formulate the modal soundness result for **GL**.

   (c) Formulate the arithmetical soundness result for **GL**.

   (d) Show how Gödel's Second Incompleteness Theorem follows from Löb's Rule.

   (e) Consider a fixpoint λ of $\neg\mathsf{Bew}_{\mathsf{PA}}(x)$. State what it means that λ is a fixpoint of this particular sentence.

   (f) We say that PA is Σ_1-complete. What does this mean?

   (g) Formulate provable Σ_1-completeness.

   (h) Which modal principle reflects provable Σ_1-completeness?

   (i) Show that Σ_1-soundness is stronger than consistency.

   (j) Assume that PA is Σ_1-sound. Consider the sentence λ from Item 5e. Give proofs for the following two assertions and tell which of the two assertions can be proven using consistency of PA rather than the stronger Σ_1-soundness.

      i. $\mathsf{PA} \nvdash \lambda$.

      ii. $\mathsf{PA} \nvdash \neg\lambda$.

Day 2: Completeness results for GL

1. (a) Exhibit a **GL** frame with an increasing chain of length $\omega \cdot 2 + 2$.
   (b) Exhibit a rooted tree where each branch is of finite length but so that there is a point x with $\mathsf{Ord}(x) = \omega$.
   (c) Exhibit a rooted tree where each branch is of finite length but so that there is a point x with $\mathsf{Ord}(x) = \omega \cdot 2$.
   (d) Let FRT be the class of ordinals such that $\alpha \in \mathsf{FRT}$ iff there is some rooted tree T where each branch is of finite length and for some $x \in T$ we have $\mathsf{Ord}(x) = \alpha$. Show that FRT is closed under $\alpha \mapsto \alpha + 1$.
   (e) Show that FRT is closed under addition. That is, if $\alpha \in \mathsf{FRT}$ and $\beta \in \mathsf{FRT}$, then $\alpha + \beta \in \mathsf{FRT}$.
   (f) Show that FRT defines an initial segment of the ordinals, that is, $\alpha \in \mathsf{FRT}$ and $\beta < \alpha$ implies $\beta \in \mathsf{FRT}$.
   (g) Show that FRT is closed under unions.
   (h) Conclude that $\mathsf{FRT} = \mathsf{Ord}$.
   (i) * Determine the size of FRT^ω which defined just as FRT but now we require that at each node in the tree only countably many bifurcations/children are allowed.

2. Use the modal completeness theorem to prove

$$\mathbf{GL} \vdash \Box A \quad \Rightarrow \quad \mathbf{GL} \vdash A.$$

3. Prove the generalized fixpoint Lemma:

   Lemma 0.1. *If $\psi_1(\vec{x}), \dots \psi_n(\vec{x})$ are arithmetical formulas where the variables $\vec{x} = \langle x_1, \dots, x_n \rangle$ appear free, then there are formulas $\psi_1, \dots \psi_n$ such that, for all $i \leq n$,*

$$\mathsf{PA} \vdash \phi_i \leftrightarrow \psi_i(\ulcorner \dot{\phi_1} \urcorner, \dots, \ulcorner \dot{\phi_n} \urcorner).$$

   Hint: Use the standard fixpoint lemma and induction on n.

4. With notation as in the proof of Solovay's theorem, show that if f is an arithmetical interpretation with $f(p) = \bigvee_{w \in V(p)}$ and $w \neq 0$ then $\mathfrak{M}, w \models \phi$ if and only if $\mathsf{PA} + \theta(w) \vdash f(\phi)$.

5. In the proof of Solovay's theorem, we defined the formulas $\theta(w)$ using the multiple fixpoint lemma. Write down explicitly the fixpoint equations in terms of the provability predicate $\mathtt{prv}_T$ and the variables $\theta(w), \ulcorner \theta(w) \urcorner$.

6. **Solovay's Completeness Result**
   For each $m \in \mathbb{N}$ we consider the sentence

$$\varphi_m := \mathsf{Con}^{\mathsf{m}}_{\mathsf{PA}}(\ulcorner 1 = 1 \urcorner) \wedge \mathsf{Bew}^{\mathsf{m}+1}_{\mathsf{PA}}(\ulcorner 0 = 1 \urcorner)$$

   where we define $\mathsf{Con}^{0}_{\mathsf{PA}}(\ulcorner 1 = 1 \urcorner)$ to be just $1 = 1$.

(a) Prove that for $l \neq m$ we have that

$$\mathsf{PA} \vdash \varphi_l \to \neg\varphi_m.$$

(b) Prove that for each l,

$$\mathsf{PA} \vdash \varphi_l \to \mathsf{Bew}_{\mathsf{PA}}(\ulcorner \bigvee_{m<l} \varphi_m \urcorner).$$

(c) Prove that for $m < l$,

$$\mathsf{PA} \vdash \varphi_l \to \mathsf{Con}_{\mathsf{PA}}(\ulcorner \varphi_m \urcorner).$$

(d) Suppose that some modal formula A is not a theorem of **GL** and is refutable on a linear model. We label the top-node of this model by 0, the node immediately below that 1, etc. We now define an arithmetical translation $*$ as follows

$$p^* := \bigvee_{m \Vdash p} \varphi_m.$$

Prove that for modal formulas B we have that

$$m \Vdash B \;\Rightarrow\; \mathsf{PA} \vdash \varphi_m \to B^*,$$

and

$$m \nVdash B \;\Rightarrow\; \mathsf{PA} \vdash \varphi_m \to \neg B^*.$$

(e) Prove that for each natural number n in **GL** we can prove

$$\Diamond\Diamond^n\top \to \Diamond(\Diamond^n\top \wedge \Box^{n+1}\bot).$$

(f) Prove that for each m,

$$\mathbb{N} \models \mathsf{Con}_{\mathsf{PA}}(\ulcorner \varphi_m \urcorner).$$

(g) Prove that

$$\mathsf{PA} \vdash \varphi_n \to \neg B^* \quad \Rightarrow \quad \mathsf{PA} \vdash \mathsf{Con}_{\mathsf{PA}}(\ulcorner \varphi_n \urcorner) \to \neg\mathsf{Bew}_{\mathsf{PA}}(\ulcorner B^* \urcorner).$$

(h) Suppose that some modal formula A is not a theorem of **GL** and is refutable on a linear model. Prove that there is an arithmetical realization $*$ that maps propositional variables to Boolean combinations of sentences of the form $\mathsf{Con}^{\mathsf{m}}_{\mathsf{PA}}(\ulcorner 1 = 1 \urcorner)$ so that

$$\mathsf{PA} \nvdash A^*.$$

(i) Provide an arithmetical sentence ψ so that

$$\mathsf{PA} \nvdash \mathsf{Bew}(\ulcorner \psi \urcorner) \vee \mathsf{Bew}(\ulcorner \neg\psi \urcorner).$$

Day 3: Polymodal logics

1. Prove that $0 + \beta = \beta$ for any ordinal β.

2. Suppose we define exponentiation as

   - $\alpha^0 = 1$;
   - $\alpha^{(\beta+1)} = \alpha \times \alpha^\beta$;
   - $\alpha^\lambda = \cup_{\beta<\lambda}(\alpha^\lambda) \qquad \text{for } \lambda \in \mathsf{Lim}.$

   Is this definition equivalent to the one given in the slides?

3. Prove that if X is a scattered space then $X \models \Box(\Box p \to p) \to \Box p$.

4. Prove that if $\langle X, \mathcal{T}\rangle$ is a scattered space and $\mathcal{T} \subseteq \mathcal{S}$ then $\langle X, \mathcal{S}\rangle$ is scattered as well.

Day 4: The closed fragment

1. Show that if ϕ, ψ are formulas then $\mathfrak{Ig} \models \Box((\Box\phi \to \Box\psi) \vee (\Box\psi \to \Box\phi))$.

2. Define a valuation V on $\mathfrak{I}g$ such that $\mathfrak{Ig} \models [0]p \wedge \neg[1]p$.

   Conclude that $\mathfrak{Ic}$ is not a valid frame for the full logic GLP_ω.

3. Define a valuation V on $\mathfrak{Ic}$ such that $\mathfrak{Ic} \models \langle 0\rangle p \wedge \neg[1]\langle 0\rangle p$.

Day 5: Ordinal analysis

1. Compute the following order-types:

   (a) $o(1)$
   (b) $o(212)$
   (c) $o(100\omega)$
   (d) $o(012101210)$