

Provability logics and applications

Day 1: Provability as modality

1. Give formal proofs to the extend that $\mathbf{K} \vdash \Box A \wedge \Box B \leftrightarrow \Box(A \wedge B)$. (Hints are in the slides.)
2. Let Löb's rule –we write LR– be $\Box A \rightarrow A/A$.
 - (a) Show that $\mathbf{K} + \text{LR} = \mathbf{K}$
 - (b) Show that $\mathbf{K4} + \text{LR} = \mathbf{GL}$
3. Show that $\mathbf{GL} \vdash \Box A \rightarrow \Box\Box A$. (Hints are in the slides.)
4. Let λ be Gödel's liar sentence so that $\text{PA} \vdash \neg \text{Prv}_{\text{PA}}(\lambda) \leftrightarrow \lambda$
 - (a) Show that $\text{PA} \vdash \lambda \leftrightarrow \text{Con}_{\text{PA}}$.
 - (b) Show that if PA is consistent, then $\text{PA} \not\vdash \lambda$.

5. Löb's Theorem

In the lecture we have proven Löb's Theorem by considering a fixpoint of

$$\text{Bew}_{\text{PA}}(x) \rightarrow A.$$

- (a) Formulate Gödel's Second Incompleteness Theorem.
- (b) Formulate the modal soundness result for \mathbf{GL} .
- (c) Formulate the arithmetical soundness result for \mathbf{GL} .
- (d) Show how Gödel's Second Incompleteness Theorem follows from Löb's Rule.
- (e) Consider a fixpoint λ of $\neg \text{Bew}_{\text{PA}}(x)$. State what it means that λ is a fixpoint of this particular sentence.
- (f) We say that PA is Σ_1 -complete. What does this mean?
- (g) Formulate provable Σ_1 -completeness.
- (h) Which modal principle reflects provable Σ_1 -completeness?
- (i) Show that Σ_1 -soundness is stronger than consistency.
- (j) Assume that PA is Σ_1 -sound. Consider the sentence λ from Item 5e. Give proofs for the following two assertions and tell which of the two assertions can be proven using consistency of PA rather than the stronger Σ_1 -soundness.
 - i. $\text{PA} \not\vdash \lambda$.
 - ii. $\text{PA} \not\vdash \neg\lambda$.

Day 2: Completeness results for GL

1. (a) Exhibit a **GL** frame with an increasing chain of length $\omega \cdot 2 + 2$.
 - (b) Exhibit a rooted tree where each branch is of finite length but so that there is a point x with $\text{Ord}(x) = \omega$.
 - (c) Exhibit a rooted tree where each branch is of finite length but so that there is a point x with $\text{Ord}(x) = \omega \cdot 2$.
 - (d) Let **FRT** be the class of ordinals such that $\alpha \in \text{FRT}$ iff there is some rooted tree T where each branch is of finite length and for some $x \in T$ we have $\text{Ord}(x) = \alpha$. Show that **FRT** is closed under $\alpha \mapsto \alpha + 1$.
 - (e) Show that **FRT** is closed under addition. That is, if $\alpha \in \text{FRT}$ and $\beta \in \text{FRT}$, then $\alpha + \beta \in \text{FRT}$.
 - (f) Show that **FRT** defines an initial segment of the ordinals, that is, $\alpha \in \text{FRT}$ and $\beta < \alpha$ implies $\beta \in \text{FRT}$.
 - (g) Show that **FRT** is closed under unions.
 - (h) Conclude that $\text{FRT} = \text{Ord}$.
 - (i) * Determine the size of FRT^ω which defined just as **FRT** but now we require that at each node in the tree only countably many bifurcations/children are allowed.
2. Use the modal completeness theorem to prove

$$\mathbf{GL} \vdash \Box A \Rightarrow \mathbf{GL} \vdash A.$$

3. Prove the generalized fixpoint Lemma:

Lemma 0.1. *If $\psi_1(\vec{x}), \dots, \psi_n(\vec{x})$ are arithmetical formulas where the variables $\vec{x} = \langle x_1, \dots, x_n \rangle$ appear free, then there are formulas ϕ_1, \dots, ϕ_n such that, for all $i \leq n$,*

$$\text{PA} \vdash \phi_i \leftrightarrow \psi_i(\ulcorner \dot{\phi}_1 \urcorner, \dots, \ulcorner \dot{\phi}_n \urcorner).$$

Hint: Use the standard fixpoint lemma and induction on n .

4. With notation as in the proof of Solovay's theorem, show that if f is an arithmetical interpretation with $f(p) = \bigvee_{w \in V(p)} w$ and $w \neq 0$ then $\mathfrak{M}, w \models \phi$ if and only if $\text{PA} + \theta(w) \vdash f(\phi)$.
5. In the proof of Solovay's theorem, we defined the formulas $\theta(w)$ using the multiple fixpoint lemma. Write down explicitly the fixpoint equations in terms of the provability predicate prv_T and the variables $\theta(w), \ulcorner \theta(w) \urcorner$.

6. Solovay's Completeness Result

For each $m \in \mathbb{N}$ we consider the sentence

$$\varphi_m := \text{Con}_{\text{PA}}^m(\ulcorner 1 = 1 \urcorner) \wedge \text{Bew}_{\text{PA}}^{m+1}(\ulcorner 0 = 1 \urcorner)$$

where we define $\text{Con}_{\text{PA}}^0(\ulcorner 1 = 1 \urcorner)$ to be just $1 = 1$.

(a) Prove that for $l \neq m$ we have that

$$\text{PA} \vdash \varphi_l \rightarrow \neg \varphi_m.$$

(b) Prove that for each l ,

$$\text{PA} \vdash \varphi_l \rightarrow \text{Bew}_{\text{PA}}(\bigvee_{m < l} \varphi_m \neg).$$

(c) Prove that for $m < l$,

$$\text{PA} \vdash \varphi_l \rightarrow \text{Con}_{\text{PA}}(\neg \varphi_m \neg).$$

(d) Suppose that some modal formula A is not a theorem of **GL** and is refutable on a linear model. We label the top-node of this model by 0, the node immediately below that 1, etc. We now define an arithmetical translation $*$ as follows

$$p^* := \bigvee_{m \Vdash p} \varphi_m.$$

Prove that for modal formulas B we have that

$$m \Vdash B \Rightarrow \text{PA} \vdash \varphi_m \rightarrow B^*,$$

and

$$m \not\Vdash B \Rightarrow \text{PA} \vdash \varphi_m \rightarrow \neg B^*.$$

(e) Prove that for each natural number n in **GL** we can prove

$$\diamond \diamond^n \top \rightarrow \diamond(\diamond^n \top \wedge \square^{n+1} \perp).$$

(f) Prove that for each m ,

$$\mathbb{N} \models \text{Con}_{\text{PA}}(\neg \varphi_m \neg).$$

(g) Prove that

$$\text{PA} \vdash \varphi_n \rightarrow \neg B^* \Rightarrow \text{PA} \vdash \text{Con}_{\text{PA}}(\neg \varphi_n \neg) \rightarrow \neg \text{Bew}_{\text{PA}}(\neg B^* \neg).$$

(h) Suppose that some modal formula A is not a theorem of **GL** and is refutable on a linear model. Prove that there is an arithmetical realization $*$ that maps propositional variables to Boolean combinations of sentences of the form $\text{Con}_{\text{PA}}^m(\neg 1 = 1 \neg)$ so that

$$\text{PA} \not\vdash A^*.$$

(i) Provide an arithmetical sentence ψ so that

$$\text{PA} \not\vdash \text{Bew}(\neg \psi \neg) \vee \text{Bew}(\neg \neg \psi \neg).$$

Day 3: Polymodal logics

1. Show that $\mathbf{K} \vdash \Diamond A \wedge \Box(B) \rightarrow \Diamond(A \wedge B)$.
2. Let $A \in \mathbf{Worm}_{\alpha+1}$ and B be an arbitrary worm. Show that

$$\mathbf{GLP} \vdash (A \wedge \langle \alpha \rangle B) \leftrightarrow A\alpha B.$$

3. Show that for any natural number n ,

$$\mathbf{GLP} \vdash \langle 1 \rangle \langle 0 \rangle \langle 1 \rangle \top \rightarrow \langle 0 \rangle^n \langle 1 \rangle \top.$$

4. Prove that any topological space validates the \mathbf{K} axiom $\Box(\phi \rightarrow \psi) \rightarrow (\Box\phi \rightarrow \Box\psi)$.
5. Prove that if X is a topological space and $X \models \Box(\Box p \rightarrow p) \rightarrow \Box p$ then X is scattered.
6. Prove that if $\langle X, \mathcal{T}_0, \mathcal{T}_1 \rangle$ is a bitopological space, then $X \models [0]p \rightarrow [1]p$ if and only if $\mathcal{T}_0 \subseteq \mathcal{T}_1$.
7. Prove that if $\langle X, \mathcal{T}_0, \mathcal{T}_1 \rangle$ is a bitopological space, then $X \models \langle 0 \rangle p \rightarrow [1] \langle 0 \rangle p$ if and only if whenever $A \subseteq X$, $d_0 A$ is open in \mathcal{T}_1 .
8. Prove that if $\langle X, \mathcal{T} \rangle$ is a scattered space and $\mathcal{T} \subseteq \mathcal{S}$ then $\langle X, \mathcal{S} \rangle$ is scattered as well.

Day 4: The closed fragment

1. Prove that $0 + \beta = \beta$ for any ordinal β .
2. Suppose we define exponentiation as

- $\alpha^0 = 1$;
- $\alpha^{(\beta+1)} = \alpha \times \alpha^\beta$;
- $\alpha^\lambda = \bigcup_{\beta < \lambda} (\alpha^\beta)$ for $\lambda \in \text{Lim}$.

Is this definition equivalent to the one given in the slides?

3. Show that if $\alpha < \gamma$ and $\beta \leq \ell\gamma$ then $\alpha + \omega^\beta \leq \gamma$.
4. Show that if $\alpha < \gamma$ and $\beta < \ell\gamma$ then $\alpha + \omega^\beta < \gamma$.
5. Show that if ϕ, ψ are formulas then $\mathfrak{Jg} \models \Box((\Box\phi \rightarrow \Box\psi) \vee (\Box\psi \rightarrow \Box\phi))$.
6. Define a valuation V on \mathfrak{Jg} such that $\mathfrak{Jg} \models [0]p \wedge \neg[1]p$.
Conclude that \mathfrak{Jc} is not a valid frame for the full logic \mathbf{GLP}_ω .
7. Define a valuation V on \mathfrak{Jc} such that $\mathfrak{Jc} \models \langle 0 \rangle p \wedge \neg[1] \langle 0 \rangle p$.
8. Prove that if $f \in D$ and r is a simple sequence such that $r \sqsubseteq f$ and $r(M) < f(M)$, then for all $m < M$, $r(m) < f(m)$ as well.

Day 5: Ordinal analysis

1. For each of the following worms A , compute $o(A)$ and $\|A\|$:
 - (a) 1
 - (b) 212
 - (c) 100ω
 - (d) 012101210