An Unprovable Ramsey-type theorem

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Historical Background

- **1900** - Hilbert Program
  - 2nd question: “ Can we prove that arithmetic is consistent and free from any internal contradictions?”

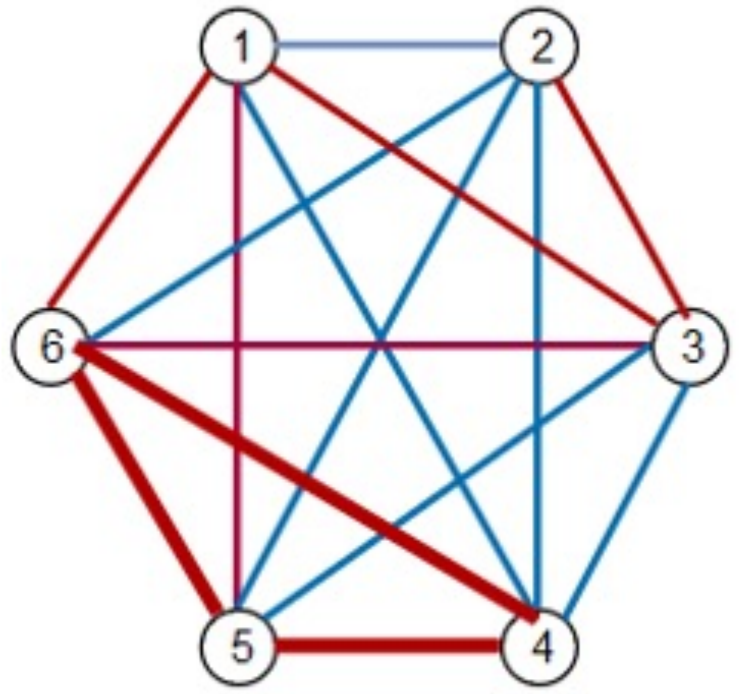
- **1931**- Gödel’s Incompleteness theorems
  - First Incompleteness theorem: In a consistent formal theory T, there exist sentences expressible in the system such that ϕ nor $\neg\phi$ is provable in T.

- **1977** - Paris-Harrington theorem
  - The first natural example of Gödel’s result. A slight variant of Finite Ramsey Theorem.

Ramsey Theory

- **Finite Ramsey's Theorem**. (**FRT**) For any natural numbers p, k, n there exists a natural number N such that if the set $[N]^p$ of all p-element subsets of the set {1,...,N} is colored with k colors then there exists a subset Y, such that for all p-element subsets of Y are monochromatic.

  Ramsey function: $R(p, k, n) = N$



- **Ramsey's Theorem.(RT)** For any natural numbers p, k, n, if $[\mathbb{N}]^p$ is colored with k colors, then there exists an infinite set $H \subset \mathbb{N}$ such that $[H]^p$ is monochromatic.

Paris Harrington Principle

- **Largeness Condition**. A set $S \subseteq \mathbb{N}$ is relatively large if $card(S) \geq min(S)$.

  $\{3,15,34,58\}$ is large but $\{4,45,624\}$ is not.

- **Strengthened Finite Ramsey Theorem**. **(FRT*)** For all natural numbers p, k, n there exists an integer N such that if $[n,N]^p$ is k-colored, there exists a relatively large homogeneous subset Y of {n,...,N} and $|Y| \geq minY$.

  The modified Ramsey function: $R*(p,k,n) = N$.

- **Paris-Harrington Theorem.** FRT* is not provable in PA.

  Loebl-Nešetril proof

Definitions and Properties

- **Countable Ordinals.**
- **Cantor Normal Form.** Each ordinal $0 < \alpha < \varepsilon_0$ has a unique representation

$$\alpha = \omega^{\alpha_1} \cdot n_1 + \ldots + \omega^{\alpha_t} \cdot n_t$$

where $\alpha > \alpha_1 > \ldots > \alpha_t$ and $n_1, \ldots, n_t > 0$.

- For every $i \leq t$ we define:

$$S_i(\alpha) = \omega^{\alpha_i} \cdot n_i$$

$$E_i(\alpha) = \alpha_i$$

$$K_i(\alpha) = n_i$$

Definitions and Properties

- **Fundamental sequence**

  Let $\alpha = \omega^{\alpha_1} \cdot n_1 + \ldots + \omega^{\alpha_t} \cdot (n_t + 1)$
  Then a standard assignment of fundamental sequences to countable ordinals is defined as:

  $\alpha[k] = \omega^{\alpha_1} \cdot n_1 + \ldots + \omega^{\alpha_t} \cdot n_t + \omega^{\alpha_t}[k]$

  If $\alpha = \omega^{\beta+1}$ then $\alpha[k] = \omega^{\beta} \cdot k$

  If $\alpha = \omega^{\beta}$ where β is a limit ordinal then $\alpha[k] = \omega^{\beta[k]}$

  Let $(\alpha + 1)[k] = \alpha$ and $0[k] = 0$

Definitions and Properties

- **Hardy Hierarchy.** For $\alpha \leq \varepsilon_0$ let:

  $H_0(x) = x$

  $H_{\alpha+1}(x) = H_\alpha(x+1)$

  $H_\alpha(x) = H_{\alpha[x]}(x)$ in case α is a limit ordinal.

- **Proposition**. If $m < n$ then $H_\alpha(m) < H_\alpha(n)$.

- **Theorem (Wainer).** *Let f be a provably total recursive function in PA. Then there exists an $\alpha < \varepsilon_0$ such that $f(n) < H_\alpha(n)$.*

Definitions and Properties

- **Height** of an ordinal α is defined as $h(\alpha) = min(h : \alpha < \omega_h)$ where $\omega_h = \omega^{\omega^{\cdots \omega}}\}h$-times.

  $h(\alpha) = h$.

  The height of the exponents of α's normal form is $h(\alpha_i) \leq h(\alpha) - 1$.

- **Rank** is defined inductively as

$$r(\alpha) = \alpha \text{ , for } \alpha \text{ a natural number,}$$

$$r(\alpha) = max\{n_1, \ldots, n_t, t, r(\alpha_1), \ldots, r(\alpha_t)\} \text{ , otherwise.}$$

Definitions and Properties

- **Definition** (Good Couple): *A good couple is a pair* (α, p) *where* $\alpha < \varepsilon_0$ *and* $p > r(\alpha) + h(\alpha)$.
- **Proposition:** $r(\alpha[n]) \leq max\{r(\alpha), n\}$

  Proof. By induction on α.

  \- $\alpha = \beta + 1, \beta + 1[k] = \beta$ so $r(\beta) \leq max\{r(\beta + 1), n\}$

  \- α a limit ordinal, $\alpha[n]$ is either a limit or successor ordinal.

  If $\alpha = \omega^{\beta+1}$, $r(\omega^{\beta} \cdot n) = max\{r(\beta), n\} \leq max\{r(\omega^{\beta+1}), n\}$.

  If $\alpha = \omega^{\beta}$, for β limit ordinal, $r(\omega^{\beta[n]}) = max\{r(\beta[n]), n\} \leq max\{r(\omega^{\beta}), n\}$

- **Definition**: Let (α, p) be a good couple. Then define $(\alpha, p)^+$ as:

  $(\alpha + 1, p)^+ = (\alpha, p + 1)$

  $(\alpha, p)^+ = (\alpha[p - h(\alpha]), p + 1)$ when α limit ordinal.

  From Proposition, $r(\alpha[p - h(\alpha)]) \leq max\{r(\alpha), p - h(\alpha)\}$, so every pair defined this way is a good couple as well.

Definitions and Properties

- **Definition**: *Let* $(\alpha, p + h(\alpha) + 1)$ *be a good couple.*

  A good system $L(\alpha, p)$ *is generated by iterating the function* $()^+$ *on this couple till the first coordinate becomes zero. The length of this system is denoted by* $l(\alpha, p) = |L(\alpha, p)|$.

  $$(\alpha, p + h(\alpha) + 1)$$

  $$(\alpha, p + h(\alpha) + 1)^+$$

  $$((\alpha, p + h(\alpha) + 1)^+)^+$$

  $$\vdots$$

  Until α *is zero.*

Definitions and Properties

- **Lemma (Long Sequence Lemma)**: *Let* $(\alpha, p + h(\alpha) + 1)$ *be a good couple. Then* $l(\alpha, p) > H_\alpha(p) - p$

  Proof: by transfinite induction

  - α is a natural number,

    $l(n, p)$ is the length of the sequence $n, p + h(\alpha) + 1 \underset{h(\alpha)=1}{=} (n, p + 2)$

    $l(\alpha, p) = n + 1 > n = H_n(p) - p$

  - α a successor ordinal

    $l(\alpha + 1, p + h(\alpha + 1) + 1) = 1 + l(\alpha, p + h(\alpha) + 2) \underset{\text{by IH}}{>} 1 + H_\alpha(p + 1) - p - 1 = H_{\alpha+1}(p) - p$

    Note. $l(\alpha + 1, p) = 1 + l(\alpha, p + 1)$

  - α a limit ordinal

    $l(\alpha, p + h(\alpha) + 1) > H_\alpha(p) - p$

Definitions and Properties

- Long Sequence lemma result can be extended to ω_h as $l(\omega_h, p) > H_{\omega_h}(p) - p$

- **Definition.** A good system of height $\leq h$ is a good system where all ordinals have height $\leq h$. $L(\omega_{h-1}, p)$ is a good system of height $\leq h$.

Unprovability of PH

- Let A be a good system, then we pick an x-element subset of A, $T = \{(\beta_1, q_1), ..., (\beta_x, q_x)\}$.
- **Definition.** A set T is a right x-set of A if ordinals $\beta_1, ..., \beta_x$ are pairwise distinct and for $i > j$, $\beta_i > \beta_j$ iff $q_i < q_j$.

  Because of the way system $A = L(\omega_{h-1}, p)$ has been defined, all x-sets are right.

- **Definition**. [(x, y)-Paris coloring]
- A coloring of all the right x-sets of a good system by y colors is called an (x, y)-Paris coloring if there is **not** a subsystem T' of A, such that $T' = (\beta_1, q_1), ..., (\beta_m, q_m)$, with the properties,

  - All x-sets in T' are right and receive the same color

  - $m \geq min\{q_1, ..., q_m\}$

- **Coloring Lemma**

  Let A be a good system of height $\leq h$, for $h \geq 2$. Assume that $h + 1 < q$ for all $(\alpha, q) \in A$. Then

  $\forall y \geq 3^{(h+1)^2+1}$ there exists a $(h + 1, y)$-Paris coloring of A.

Unprovability of PH

- **How the coloring lemma implies the unprovability of PH:**
- **Corollary**
  For every $h \geq 2$,
  $$R^*(h+1, 3^{(h+1)^2+1}, 2h+1) \geq H_{\omega_{h-1}}(2h+1) - 2h - 1$$
  where $R^*(p, k, n) = min\{N \stackrel{*}{\rightarrow} (n)^k_r\}$ is the Ramsey number for the PH-version.
- **Observation**. $L(\omega_{h-1}, p) = \{(\alpha_1, p_1), \ldots, (\alpha_N, p_N)\}$ and $p_1 < \ldots < p_N$.
  By definition of the good systems $(\alpha_1, p_1) = (\omega_{h-1}, p+h+1)$ and $p_{i+1} = p_i + 1$ for $i = 1,...,N-1$.
- **Proof**
  By Long Sequence lemma, $l(\omega_{h-1}, p) = N \geq H_{\omega_{h-1}}(p) - p$.
  Let $x = h+1$ and $y = 3^{(h+1)^2+1}$.
  By Coloring lemma, there exists an (x, y)-Paris coloring of $L(\omega_{h-1}, p)$ whose all $(h+1)$-sets are good by definition.
  Then this coloring induces a coloring of the set of all $(h+1)$-sets in the set $\{p_1, \ldots, N\}$.
  So by definition of (x, y)-coloring, $R^*(h+1, y, p_1) > N$.
  Let $p = h$, $R^*(h+1, 3^{(h+1)^2+1}, 2h+1) \geq H_{\omega_{h-1}}(h) - h$.

Unprovability of PH

- **Definition**. Let $\beta < \alpha < \epsilon_0$, set

$$d(\alpha, \beta) = min\{i : S_i(\alpha) \neq S_i(\beta)\}$$

$$K(\alpha, \beta) = K_{d(\alpha,\beta)}(\alpha)$$

$$E(\alpha, \beta) = E_{d(\alpha,\beta)}(\alpha)$$

- **Lemma**. For $\alpha > \beta > \gamma$, let $d(\alpha, \beta) \leq d(\beta, \gamma)$ and $K(\alpha, \beta) \leq K(\beta, \gamma)$ then $E(\alpha, \beta) > E(\beta, \gamma)$

  Proof.

  - $d(\alpha, \beta) = d(\beta, \gamma) = i$

    $S_i(\alpha) > S_i(\beta)$ and $K_i(\alpha) \leq K_i(\beta)$ so it must be that $E_i(\alpha) > E_i(\beta)$ otherwise $\alpha < \beta$.

  - $d(\alpha, \beta) = i$ and $d(\beta, \gamma) = j$

    If i<j then $E_i(\alpha) \geq E_i(\beta) > E_j(\beta)$

Unprovability of PH

- **Definition**. For an m-set $\beta_1 > \beta_2 > \ldots > \beta_m$, $m \geq 3$, define the shift vector $v = (v_1, \ldots, v_{m-2})$ where each v_i is assigned a color under $\chi_3()$:

$$\chi_3(\{\beta_1, \beta_2, \beta_3\}) = \begin{cases} \nearrow, & \textit{if } d(\beta_1, \beta_2) > d(\beta_2, \beta_3). \\ \uparrow, & \textit{if } d(\beta_1, \beta_2) \leq d(\beta_2, \beta_3) \wedge K(\beta_1, \beta_2) > K(\beta_2, \beta_3). \\ \downarrow, & \textit{otherwise.} \end{cases}$$

- Example.

$$\alpha_1 = \omega^7 \cdot 5 + \omega^6$$
$$\alpha_2 = \omega^7 \cdot 4 + \omega^4 \cdot 3$$
$$\alpha_3 = \omega^7 \cdot 4 + \omega^4$$
$$\alpha_4 = \omega^7 + \omega^5 + \omega^3 \cdot 4$$
$$\alpha_5 = \omega^7 + \omega^5 + \omega \cdot 3 + 5$$

Unprovability of PH

- **Proof of the Coloring Lemma.** By induction on h.
- Base case, $h = 2$.
  - Assign to each triple $(\beta_1, \beta_2, \beta_3)$ a color under $\chi()$
  - Prove this is a $(3{,}3)$-Paris coloring
  - Let the set $T = (\beta_1, q_1), \ldots, (\beta_m, q_m)$ be a subsystem of A such that each of its triples is right and monochromatic.
  - $|T| = m$, by assumption $h + 1 < q_1 = min(q_1, \ldots, q_m)$

- $\chi_3(T) = \nearrow$

  $d(\beta_1, \beta_2) = i + 1$ and $d(\beta_2, \beta_3) = i$

  β_1 needs to have at least 2 terms.

  $m \leq 1 + max\{t : S_t(\beta_1) \neq 0)\} < q_1$

Unprovability of PH

- **Proof of the Coloring Lemma.** By induction on h.
- Base case, $h = 2$.
- $\chi_3(T) = \uparrow$

  $d(\beta_1, \beta_2) = i$ and $d(\beta_2, \beta_3) = i + 1$, $K_i(\beta_1) > K_{i+1}(\beta_2)$

  $K_i(\beta_1)$ needs to be at least 2.

  $m \leq 1 + K_i(\beta_1) < q_1$
- $\chi_3(T) = \downarrow$

  ,$d(\beta_1, \beta_2) = i$ and $d(\beta_2, \beta_3) \geq i + 1$, $E_i(\beta_1) > E_{i+1}(\beta_2)$

  So $E_i(\beta_1)$ must be at least 2.

  $m \leq 1 + E_i(\beta_1) < q_1$

Unprovability of PH

- **Induction step.** Assume for h, prove for h+1
  - Let $(\beta_1, q_1), \ldots, (\beta_{h+2}, q_{h+2})$
  - By IH: v_1, for $(h+1)$-tuple $(\beta_1, q_1), \ldots, (\beta_{h+1}, q_{h+1})$
    v_2 for $(h+1)$-tuple $(\beta_2, q_2), \ldots, (\beta_{h+2}, q_{h+2})$.
  - Define a new color assignment for $(h+2)$-tuples, $\chi_{(h+2)}()$:

$$\chi_{(h+2)}((\beta_1, q_1), ..., (\beta_{h+2}, q_{h+2})) = \begin{cases} \chi_{(h+1)}((E_1(\beta_1), s_1), ..., (E_{h+1}(\beta_{h+1}), s_{h+1}))), \\ \qquad \text{if } v_1 = v_2 = \downarrow, \text{ where } s_i = q_i - 1 \ . \\ (v_1, v_2), \\ \qquad \text{otherwise} \end{cases}$$

Unprovability of PH

- **Induction step.** Assume for h, prove for h+1
  - Let $T = (\beta_1, q_1), ..., (\beta_m, q_m)$ be monochromatic under $\chi_{(h+2)}()$.
  - Proof this is a $(h+2, y)$ - Paris coloring.

- $\chi(T) = (v_1, v_2)$ and $v_1 \neq v_2$.

  $m \leq h+2$ and $q_1 = min(q_1, ..., q_m) > h+1$ thus $m < q_1$

- $\chi(T) = (v_1, v_2)$ and $v_1 = v_2 \in \{ \nearrow, \uparrow \}$

  as for $h = 2$, $m < q_1$

- $\chi(T) = (v_1, v_2)$and $v_1 = v_2 = \downarrow$

  The system $(E_1(\beta_1), s_1), ..., (E_{h+1}(\beta_{h+1}), s_{h+1})$ is a right $(h+1)$-set and monochromatic by definition.

  By IH, $m - 1 \leq s_1 = min(s_1, ..., s_{m-1}) = q_1 - 1$

  $m < q_1 = min(q_1, ..., q_m)$

Remarks

- PH principle can be restricted to 2 colors producing a stronger unprovability result .
- **Largeness Condition**. A set $X \subseteq \mathbb{N}$ is n-large, where $n \in \mathbb{N}$, if X has at least n elements.
- α**-largeness**. A set X is ω-large if $X \backslash \{\min X\}$ is min X-large; X has strictly more than min X elements.