A Modular Curry-Howard-Lambek Correspondence for Belief and Knowledge

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Italiadomani
PIANO NAZIONALE DI RIPRESA E RESILIENZA

IMT SCUOLA ALTI STUDI LUCCA

Outline

"Setting the Stage"

Prologue
- Origins
- BHK Explanation
- Axiomatisation(s)
- Relational Semantics

Proofs
- Natural Deduction for Intuitionistic Logic
- Natural Deduction for Intuitionistic Epistemic Logics

Programs
- Curry-Howard for NJ_p
- Modal Type Theory for Epistemic States
- Normalisation Theorems

Arrows
- Curry-Howard-Lambek for NJ_p
- Categorical Structures for Intuitionistic Epistemic Logics
- Soundness and Completeness

Epilogue

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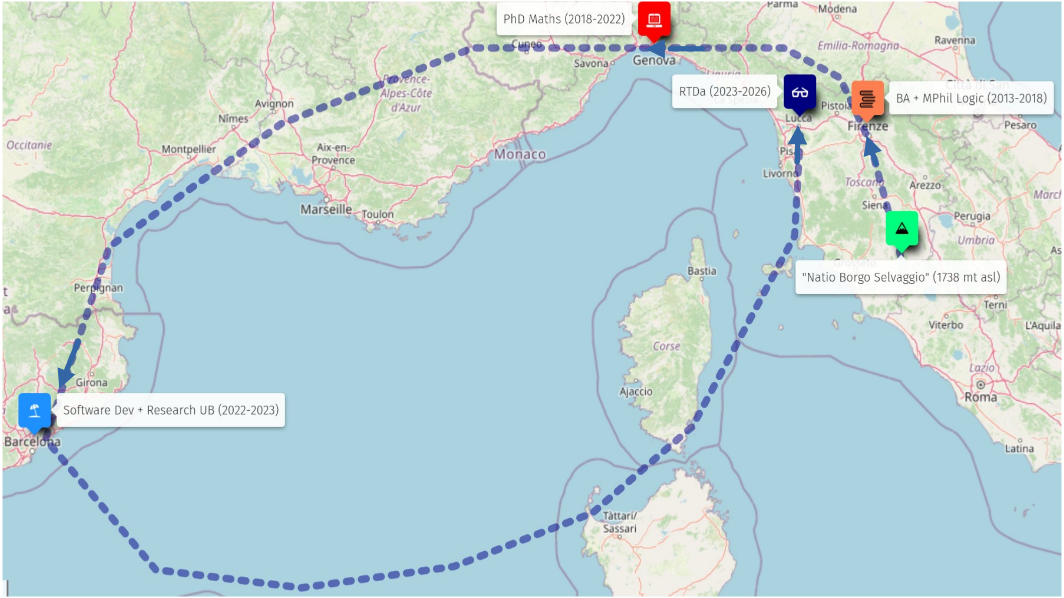
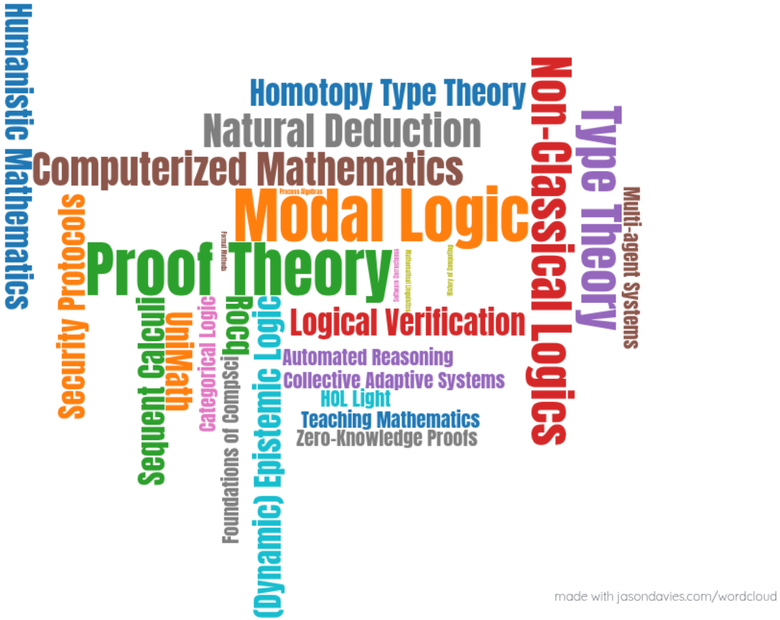
Westward Ho, and Return



Some Research Topics



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Knowledge and Verification

We adopt the view that an ***an intuitionistic epistemic state (belief or knowledge) is the result of verification*** *where a verification is evidence considered sufficiently conclusive for practical purposes.*

[AP16]

A Two-Part Example from Mathematical Practice

Part I: Verification-based Belief

Your favourite top-rated mathematician has posted a proof sketch of a conjecture A *in a very specialised mathematical field on her blog. Her informal argument is convincing, and her expertise justifies your trust in the conjecture. You are not an expert in that field, but she is; you do not have direct access to a detailed proof of* A*, but it is reasonable for you to believe that* A *does hold.*

A Two-Part Example from Mathematical Practice

Part II: Verification-based Knowledge

Some time has passed since the mathematician's post on conjecture A. You are again surfing the Net looking for new results in the field A belongs, and come across another post about that conjecture. This time, a famous top-rated computer scientist's blog announces that A has been refuted. And there's more: Her refutation has been computerised, and the formal proof is freely available on the Net. If you know the programming language at the base of her computer program to refute A – a proof assistant, or a highly specialised theorem prover – you can read the proof and see that $\neg A$ holds, so that your original trust in A is misplaced.

In Plain Terms

We can read any epistemic formula $\Box A$ as asserting that A has a proof which is not necessarily specified in the process of verification, or more generally that it is verified that A holds in some not specified *constructive* sense.

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This allows to apply verificationist/intuitionistic epistemic reasoning in various contexts which are not necessarily in the mathematical domain, such as:

- Testimony of authority;
- Zero-knowledge protocols;
- Highly probable truth.

According to the BHK semantics, a proposition A is true if there is a proof of it, and false if one can show that assuming A leads to a contradiction. More precisely:

- there is no proof of $\bot$;
- a proof p of $A \wedge B$ consists of a pair $\langle a, b \rangle$ where a is a proof of A and b is a proof of B;
- a proof p of $A \vee B$ is a pair $\langle n, q \rangle$ where $n = 0$ and q proves A, or $n = 1$ and q proves B;
- a proof p of $A \to B$ is a rule which transforms any proof q of A into a proof $p(q)$ of B.

If we add an epistemic modal operator $\Box$ to the language of intuitionistic propositional logic, we must extend the BHK interpretation to any formula $\Box A$.
The epistemic clause adopted is

- a proof p of $\Box A$ is a conclusive evidence of *verification* that A has a proof.

1. Every proof is a verification;
2. That something is a proof is itself capable of proof.

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Considering 1. and this extended semantics we have that

$$\text{Intuitionistic Truth} \Rightarrow \text{Intuitionistic Knowledge/Belief}$$

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Considering 1. and this extended semantics we have that

$$\text{Intuitionistic Truth} \Rightarrow \text{Intuitionistic Knowledge/Belief}$$

since, in general,

$$\text{it is proved that } A \text{ is verified}$$

is a *weaker statement* than

$$\text{it is proved that } A$$

$\mathbb{IEL}^-$

Axioms

1. Axioms of propositional intuitionistic logic;
2. $\Box(A \to B) \to \Box A \to \Box B$; (K-scheme)
3. $A \to \Box A$. (co-reflection)

Rules

$$\frac{A \to B \qquad A}{B}\ MP$$

$\mathbb{IEL}$

Extend $\mathbb{IEL}^-$ by *one* of the following axiom schemas:

1. $\neg A \to \neg \Box A$;
2. $\neg(\Box A \wedge \neg A)$;
3. $\neg \Box \bot$;
4. $\neg\neg(\Box A \to A)$;
5. $\Box A \to \neg\neg A$.

Lemma

(i) *Over* $\mathbb{IEL}^-$*, the axiom schemas for intuitionistic knowledge are all equivalent.*

(ii) $\mathbb{IEL}^- \nvdash \Box A \to A$.

(iii) *Over* $\mathbb{IK}$*, the axiom schemas for intuitionistic knowledge are organised according to the following hierarchy:*

$$\Box A \to A$$

$$\Downarrow$$

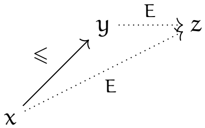
$$\neg(\Box A \wedge \neg A) \Leftrightarrow (\Box A \to \neg\neg A) \Leftrightarrow \neg\neg(\Box A \to A) \Leftrightarrow (\neg A \to \neg\Box A)$$

$$\Downarrow$$

$$\neg\Box\bot$$

A model for $\mathbb{IEL}^-$ is a quadruple $\langle W, \leqslant, v, E\rangle$ where

- $\langle W, \leqslant, v\rangle$ is a standard model for intuitionistic propositional logic;
- E is a binary 'knowledge' relation on W such that:
  - if xEy, then $x \leqslant y$; and
  - if $x \leqslant y$ and yEz, then xEz; graphically we have



- v extends to a forcing relation $\Vdash$ such that
  - $x \Vdash \Box A$ iff $y \Vdash A$ for all y such that xEy.

A model for $\mathbb{IEL}$ is a model for $\mathbb{IEL}^-$ $\langle W, \leqslant, v, E\rangle$ where the relation E satisfies the seriality condition

- for any $x \in W$, there exists a $y \in W$ such that xEy.

Lemma (Monotonicity)

For each model and a formula A, *if* $x \vDash A$ *and* $x \leqslant y$, *then* $y \vDash A$.

Lemma (Soundness)

If $\mathbb{IEL}^- \vdash A$ *then* A *holds in any* $\mathbb{IEL}^-$*-model.*

Theorem (Completeness)

If A *holds in any* $\mathbb{IEL}^-$*-model, then* $\mathbb{IEL}^- \vdash A$.

Lemma (Monotonicity)

For each model and a formula A*, if* $x \vDash A$ *and* $x \leqslant y$*, then* $y \vDash A$*.*

Lemma (Soundness)

If $\mathbb{IEL}^- \vdash A$ *then* A *holds in any* $\mathbb{IEL}^-$*-model.*

Theorem (Completeness)

If A *holds in any* $\mathbb{IEL}^-$*-model, then* $\mathbb{IEL}^- \vdash A$*.*

Proof.

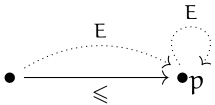
By constructing a canonical model in the standard way. ⊠

Lemma

$\mathbb{IEL}^{-} \nvdash \Box A \to A$

Proof.

Consider the following model:



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NJ

Propositional Fragment

$$\dfrac{\begin{matrix}\Gamma\\ \vdots\\ \bot\end{matrix}}{A}\ \bot_{J}$$

$$\dfrac{}{\top}\ \top_{J}$$

$$\dfrac{\begin{matrix}\Gamma\\ \vdots\\ A\end{matrix}\qquad\begin{matrix}\Delta\\ \vdots\\ B\end{matrix}}{A \wedge B}\ \wedge\mathcal{I}$$

$$\dfrac{\begin{matrix}\Gamma\\ \vdots\\ A \wedge B\end{matrix}}{A}\ \wedge\mathcal{E}_1$$

$$\dfrac{\begin{matrix}\Gamma\\ \vdots\\ A \wedge B\end{matrix}}{B}\ \wedge\mathcal{E}_2$$

$$\dfrac{\begin{matrix}\Gamma\\ \vdots\\ A\end{matrix}}{A \vee B}\ \vee\mathcal{I}_1$$

$$\dfrac{\begin{matrix}\Gamma\\ \vdots\\ B\end{matrix}}{A \vee B}\ \vee\mathcal{I}_2$$

$$\dfrac{\begin{matrix}\Gamma\\ \vdots\\ A \vee B\end{matrix}\qquad\begin{matrix}\Delta, [A]^1\\ \vdots\\ C\end{matrix}\qquad\begin{matrix}\Theta, [B]^2\\ \vdots\\ C\end{matrix}}{C}\ \vee\mathcal{E}{:}1,2$$

$$\dfrac{\begin{matrix}\Gamma, [A]^1\\ \vdots\\ B\end{matrix}}{A \to B}\ \to\mathcal{I}{:}1$$

$$\dfrac{\begin{matrix}\Gamma\\ \vdots\\ A \to B\end{matrix}\qquad\begin{matrix}\Delta\\ \vdots\\ A\end{matrix}}{B}\ \to\mathcal{E}$$

Natural Deduction for Intuitionistic Belief

IEL^{-}

Extend NJ_p by the following rule:

$$\dfrac{\begin{array}{c}\Gamma_1\\ \vdots\\ \Box A_1\end{array}\quad \vdots \quad \begin{array}{c}\Gamma_n\\ \vdots\\ \Box A_n\end{array}\quad \begin{array}{c}[A_1, \cdots, A_n], \Delta\\ \vdots\\ B\end{array}}{\Box B}\;\Box\mathcal{I}$$

where Γ and Δ are *multisets of formulas*, and $A_1, \cdots, A_n$ are *all* discharged.

Natural Deduction for Intuitionistic Knowledge

IEL

Extend $\mathsf{NJ_p}$ by the following *pair* of rules:

$$
\dfrac{\begin{array}{c}\Gamma_1\\ \vdots\\ \Box A_1\end{array} \quad \begin{array}{c}\\ \vdots\\ \end{array} \quad \begin{array}{c}\Gamma_n\\ \vdots\\ \Box A_n\end{array} \quad \begin{array}{c}[A_1, \cdots, A_n], \Delta\\ \vdots\\ B\end{array}}{\Box B}\;\Box\mathcal{I}
\qquad\qquad
\dfrac{\begin{array}{c}\Gamma\\ \vdots\\ \Box A\end{array} \quad \begin{array}{c}[A], \Delta\\ \vdots\\ \bot\end{array}}{\bot}\;\Box\mathcal{E}
$$

where: Γ and Δ are *finite multisets of formulas*; $A_1, \cdots, A_n$ are *all* discharged in $\Box\mathcal{I}$; and A is discharged by $\Box\mathcal{E}$.

Lemma

$\Gamma \vdash_{\mathsf{L}} A$ *iff* $\Gamma \vdash_{\mathbb{L}} A$.

Lemma

$\Gamma \vdash_{\mathsf{L}} A$ *iff* $\Gamma \vdash_{\mathbb{L}} A$.

Proof.

Assume $\Gamma \vdash_{\mathbb{L}} A$. We proceed by induction on the derivation.
We only need to check that the axiom schemas for the modalities can be derived in the natural deduction calculus:

- K:
$$\dfrac{\Box(A \to B) \qquad \Box A \qquad \begin{matrix}[A \to B, A] \\ \vdots \\ B\end{matrix}}{\Box B}\,\Box\mathcal{I}$$
;

- co-reflection:
$$\dfrac{\dfrac{[A]^1}{\Box A}\,\Box\mathcal{I}}{A \to \Box A}\,{\to}\mathcal{I}{:}1$$
;

- schema $\Box A \to \neg\neg A$, we have in IEL
$$\dfrac{\dfrac{\dfrac{[\Box A]^3 \qquad \dfrac{[A]^1 \qquad [\neg A]^2}{\bot}\,{\to}\mathcal{E}}{\bot}\,\Box\mathcal{E}{:}1}{\neg\neg A}\,{\to}\mathcal{I}{:}2}{\Box A \to \neg\neg A}\,{\to}\mathcal{I}{:}3$$

Lemma

$\Gamma \vdash_{\mathsf{L}} A$ *iff* $\Gamma \vdash_{\mathbb{L}} A$.

Proof.

Conversely, assume $\Gamma \vdash_{\mathsf{L}} A$. We proceed by induction on the derivation.
We only need to check that the natural deduction rules for the modalities are admissible in the axiomatic calculus, by leveraging the fact that $\mathbb{L}$ is a normal modal logic [AP16].

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Curry-Howard Correspondence

A Glossary [SU06]

Type Theory	Logic
type	proposition
term	proof
type constructor	logic connective
constructor	introduction rule
destructor	elimination rule
redex	proof detour
reduction	normalisation
normal form	normal proof
inhabitation	provability

Curry-Howard Correspondence

Proofs-as-Programs for NJ_p

$$\dfrac{\begin{array}{c}\Gamma, [A]^1\\ \vdots \\ B\end{array}}{A \to B}\ {\to}\mathcal{I}{:}1 \qquad\qquad \dfrac{\begin{array}{c}\Gamma\\ \vdots \\ A \to B\end{array} \qquad \begin{array}{c}\Delta\\ \vdots \\ A\end{array}}{B}\ {\to}\mathcal{E}$$

$$\dfrac{\Gamma \vdash t : B}{\Gamma - x : A \vdash \lambda x.t : A \to B}\ {\to}\textit{-intro} \qquad\qquad \dfrac{\Gamma \vdash t : A \to B \qquad \Delta \vdash s : A}{\Gamma, \Delta \vdash ts : B}\ {\to}\textit{-elim}$$

Curry-Howard Correspondence

Proofs-as-Programs for NJ_p

$$\dfrac{\begin{array}{c}\Gamma\\ \vdots\\ A\end{array} \qquad \begin{array}{c}\Delta\\ \vdots\\ B\end{array}}{A \wedge B}\ \wedge\mathcal{I} \qquad\qquad \dfrac{\begin{array}{c}\Gamma\\ \vdots\\ A \wedge B\end{array}}{A}\ \wedge\mathcal{E}_1 \qquad\qquad \dfrac{\begin{array}{c}\Gamma\\ \vdots\\ A \wedge B\end{array}}{B}\ \wedge\mathcal{E}_2$$

$$\dfrac{\Gamma_1 \vdash t_1 : A \qquad \Gamma_2 \vdash t_2 : B}{\Gamma_1, \Gamma_2 \vdash (t_1, t_2) : A \times B}\ \times\textit{-intro} \qquad \dfrac{\Gamma \vdash t : A \times B}{\Gamma \vdash (\pi_1 t) : A}\ \times\textit{-elim}_1 \qquad \dfrac{\Gamma \vdash t : A \times B}{\Gamma \vdash (\pi_2 t) : B}\ \times\textit{-elim}_2$$

$$\dfrac{\begin{array}{c}\Gamma\\ \vdots\\ A\end{array}}{A \vee B}\ \vee\mathcal{I}_1 \qquad \dfrac{\begin{array}{c}\Gamma\\ \vdots\\ B\end{array}}{A \vee B}\ \vee\mathcal{I}_2 \qquad \dfrac{\begin{array}{c}\Gamma\\ \vdots\\ A \vee B\end{array} \quad \begin{array}{c}\Delta, [A]^1\\ \vdots\\ C\end{array} \quad \begin{array}{c}\Theta, [B]^2\\ \vdots\\ C\end{array}}{C}\ \vee\mathcal{E}:1,2$$

$$\dfrac{\Gamma \vdash t : A}{\Gamma \vdash (\mathsf{in}_1 t) : A + B}\ +\text{-}\mathit{intro}_1 \qquad \dfrac{\Gamma \vdash s : B}{\Gamma \vdash (\mathsf{in}_2 s) : A + B}\ +\text{-}\mathit{intro}_2$$

$$\dfrac{\Gamma \vdash t : A + B \qquad \Delta, x_1 : A \vdash t_1 : C \qquad \Theta, x_2 : B \vdash t_2 : C}{\Gamma, \Delta, \Theta \vdash \mathsf{C}(t, (x_1 : A.t_1), (x_2 : B.t_2)) : C}\ +\text{-}\mathit{elim}: x_1, x_2$$

$$\dfrac{\begin{array}{c}\Gamma\\ \vdots\\ \bot\end{array}}{A}\ \bot_J \qquad\qquad \dfrac{}{\top}\ \top_J$$

$$\dfrac{\Gamma \vdash t : \bot}{\Gamma \vdash (\mathsf{ex}\, t) : A}\ \textit{exfalso} \qquad\qquad \dfrac{}{\Gamma \vdash * : \top}\ \textit{unit}$$

Curry-Howard Correspondence

Detours and Permutations in Proof-Terms

$$\dfrac{\dfrac{\begin{matrix}\vdots\\ A_1\end{matrix} \qquad \begin{matrix}\vdots\\ A_2\end{matrix}}{A_1 \wedge A_2}\,{\scriptstyle \wedge\mathcal{I}}}{A_i}\,{\scriptstyle \wedge\mathcal{E}_i} \qquad \rightsquigarrow_{\Xi} \qquad \begin{matrix}\vdots\\ A_i\end{matrix}$$

$$\dfrac{\dfrac{\begin{matrix}\vdots\\ A_i\end{matrix}}{A_1 \vee A_2}\,{\scriptstyle \vee\mathcal{I}_i} \qquad \begin{matrix}[A_1]^1\\ \vdots\\ C\end{matrix} \qquad \begin{matrix}[A_2]^2\\ \vdots\\ C\end{matrix}}{C}\,{\scriptstyle \vee\mathcal{E}:1,2} \qquad \rightsquigarrow_{\Xi} \qquad \begin{matrix}\vdots\\ A_i\\ \vdots\\ C\end{matrix}$$

$$\dfrac{\dfrac{\begin{matrix}\vdots\\ B\end{matrix}}{A \to B}\,{\scriptstyle \to\mathcal{I}} \qquad \begin{matrix}\vdots\\ A\end{matrix}}{B}\,{\scriptstyle \to\mathcal{E}} \qquad \rightsquigarrow_{\Xi} \qquad \begin{matrix}\vdots\\ B\end{matrix}$$

$$\dfrac{\dfrac{\begin{matrix}\vdots\\ A \vee B\end{matrix} \quad \begin{matrix}[A]^1\\ \vdots \\ C\end{matrix} \quad \begin{matrix}[B]^2\\ \vdots\\ C\end{matrix}}{C}\,\vee\mathcal{E}{:}1{,}2 \quad \begin{matrix}\vdots\end{matrix}}{D}\,\mathcal{E}\text{—rule} \qquad \rightsquigarrow_{\Xi}$$

$$\rightsquigarrow_{\Xi} \quad \dfrac{\begin{matrix}\vdots\\ A\vee B\end{matrix} \quad \dfrac{\begin{matrix}[A]^1\\ \vdots\\ C\end{matrix}\quad \begin{matrix}\vdots\end{matrix}}{D}\,\mathcal{E}\text{—rule} \quad \dfrac{\begin{matrix}[B]^2\\ \vdots\\ C\end{matrix}\quad \begin{matrix}\vdots\end{matrix}}{D}\,\mathcal{E}\text{—rule}}{D}\,\vee\mathcal{E}{:}1{,}2$$

$$\dfrac{\dfrac{\begin{matrix}\vdots\\ \bot\end{matrix}}{A}\,\bot_J \quad \begin{matrix}\vdots\end{matrix}}{B}\,\mathcal{E}\,/\,\textit{exfalso}\text{—rule} \qquad \rightsquigarrow_{\Xi} \qquad \dfrac{\begin{matrix}\vdots\\ \bot\end{matrix}}{B}\,\bot_J$$

Lemma

Ξ-reductions have the Church-Rosser property.

Theorem (Weak Normalization)

For every $t : A$, *there exists a unique* Ξ*-normal form.*

Theorem (Strong Normalization)

For every $t : A$, *all* Ξ*-reductions of* $t : A$ *terminate.*

Theorem (Strong normalisation, Prawitz 1965–71)

Given a deduction of A *from* Γ *every reduction based on the conversions of* Ξ *terminates in a normal deduction.*

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Given a deduction of A from Γ every reduction based on the conversions of Ξ terminates in a normal deduction.

Theorem (Subformula principle, Prawitz 1965–71)

Every formula occurring in a normal deduction is a subformula of the conclusion, or of some active assumption.

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Given a deduction of A from Γ every reduction based on the conversions of Ξ terminates in a normal deduction.

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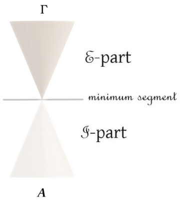
Decidability, canonicity, consistency



Applicative Programming

IEL⁻ *in Disguise [MP08]*

```ocaml
type ('a, 'e) validation_result =
  | Success of 'a
  | Failure of 'e list

type validation_error =
  | UsernameTooShort
  | InvalidEmail
  | AgeTooYoung
```

(a) Validation result and error types

```ocaml
type user = {
  username: string;
  email: string;
  age: int;
}
```

(b) User record definition

```ocaml
module ValidationApplicative = struct
  type ('a, 'e) t = ('a, 'e) validation_result
  let pure x = Success x
  let apply f_result x_result =
    match f_result, x_result with
    | Success f, Success x -> Success (f x)
    | Success _, Failure errs -> Failure errs
    | Failure errs, Success _ -> Failure errs
    | Failure errs1, Failure errs2 ->
      Failure (errs1 @ errs2)
  let (<*>) = apply
end
```

(c) Applicative instance

```ocaml
let validate_username username =
  if String.length username >= 3
  then Success username
  else Failure [UsernameTooShort]

let validate_email email =
  try
    let at_pos = String.index email '@' in
    let len = String.length email in
    if at_pos > 0 && at_pos < len - 1 &&
       not (String.contains_from email (at_pos + 1)
    ↪ '@')
    then Success email
    else Failure [InvalidEmail]
  with Not_found -> Failure [InvalidEmail]

let validate_age age =
  if age >= 13
  then Success age
  else Failure [AgeTooYoung]
```

(d) Individual validators

```ocaml
open ValidationApplicative

let create_user username email age =
  { username; email; age }

let validate_user username email age =
  pure create_user
  <*> validate_username username
  <*> validate_email email
  <*> validate_age age
```

(e) Composition of validators using applicative style

$$\frac{\Gamma_1 \vdash f_1 : \Box A_1 \qquad \cdots \qquad \Gamma_n \vdash f_n : \Box A_n \qquad x_1 : A_1, \cdots, x_n : A_n, \Delta \vdash g : B}{\Gamma_1, \cdots, \Gamma_n, \Delta \vdash (\mathsf{box}[x_1, \cdots, x_n].\, g \;\mathsf{with}\; f_1, \cdots, f_n) \;:\; \Box B}$$

$$\frac{\Gamma \vdash f : \Box A \qquad x : A, \Delta \vdash g : \bot}{\Gamma, \Delta \vdash \left(\mathsf{unbox}\, f\, \mathsf{with}\, x.g\right) : \bot}$$

Detours for Belief

Basic Cases

$$\dfrac{\dfrac{\begin{matrix}\Gamma\\ \vdots \\ \Box\vec{A}\end{matrix} \qquad \begin{matrix}[\vec{A}]^1, \vec{C}\\ \vdots \\ B\end{matrix}}{\Box B}\;\Box\mathcal{I}{:}1 \qquad \begin{matrix}\Delta\\ \vdots\\ \Box\vec{D}\end{matrix} \qquad \begin{matrix}[B,\vec{D}]^2,\vec{E}\\ \vdots\\ F\end{matrix}}{\Box F}\;\Box\mathcal{I}{:}2 \qquad \rightsquigarrow_{\rho\Box}$$

Detours for Belief

Basic Cases

$$\dfrac{\dfrac{\begin{matrix}\Gamma\\ \vdots \\ \Box\vec{A}\end{matrix} \qquad \begin{matrix}[\vec{A}]^1, \vec{C}\\ \vdots \\ B\end{matrix}}{\Box B}\ \Box\mathcal{I}{:}1 \qquad \begin{matrix}\Delta\\ \vdots\\ \Box\vec{D}\end{matrix} \qquad \begin{matrix}[B,\vec{D}]^2,\vec{E}\\ \vdots\\ F\end{matrix}}{\Box F}\ \Box\mathcal{I}{:}2 \qquad \rightsquigarrow_{\rho\Box} \qquad \dfrac{\begin{matrix}\Gamma\\ \vdots \\ \Box\vec{A}\end{matrix} \qquad \begin{matrix}\Delta\\ \vdots\\ \Box\vec{D}\end{matrix} \qquad \begin{matrix}\begin{matrix}[\vec{A}]^1, \vec{C}\\ \vdots \\ B\end{matrix} \quad [\vec{D}]^1,\vec{E}\\ \vdots\\ F\end{matrix}}{\Box F}\ \Box\mathcal{I}{:}1$$

Lemma

Deductions in IEL^{-} *strongly normalise w.r.t. the rewriting system obtained by adding* $\rho_{\Box}$ *to* Ξ.

Lemma

Deductions in IEL^- *strongly normalise w.r.t. the rewriting system obtained by adding* $\rho_\Box$ *to* Ξ.

Proof.

We define a CPS-translation $|-|$ from the λ-calculus of IEL^--deductions to typed λ-calculus with products, sums, unit and empty types:

$$\begin{array}{lcl} \ldots \\ \ldots \\ |\Box A| & := & (|A| \to q) \to q \end{array}$$

$$\begin{array}{l} \ldots \\ \ldots \\ |\mathsf{box}[x_1, \cdots, x_n].\, g \,\mathsf{with}\, f_1, \cdots, f_n| := \lambda k. |f_1|(\lambda x_1. \cdots |f_n|(\lambda x_n. k|g|) \cdots) \end{array}$$

where q is an arbitrary atom.

Lemma

Deductions in IEL^- *strongly normalise w.r.t. the rewriting system obtained by adding* $\rho_\Box$ *to* Ξ.

Proof.

$$\dfrac{\begin{matrix}\Gamma\\ \vdots\ f\\ \Box A\end{matrix} \qquad \begin{matrix}[A],\Delta\\ \vdots\ g\\ B\end{matrix}}{\Box B}\,\Box\mathcal{I} \quad \overset{|\ |}{\longmapsto} \quad \dfrac{\dfrac{\begin{matrix}|\Gamma|\\ \vdots\ |f|\\ (|A|\to q)\to q\end{matrix} \qquad \dfrac{\dfrac{[|B|\to q]^2 \qquad \begin{matrix}[|A|]^1,|\Delta|\\ \vdots\ |g|\\ |B|\end{matrix}}{q}\,\to\mathcal{E}}{|A|\to q}\,\to\mathcal{I}{:}1}{q}\,\to\mathcal{E}}{(|B|\to q)\to q}\,\to\mathcal{I}{:}2$$

To prove strong normalisation it suffices now to prove that $\rho_\Box$ is preserved by this translation, which is almost straightforward.

$\boxtimes$

$$\dfrac{\begin{matrix}\Gamma_1\\ \vdots\, f_1\\ \Box A_1\end{matrix} \quad \cdots \quad \dfrac{\begin{matrix}\vdots\, t\end{matrix} \quad \begin{matrix}\Gamma_i\\ \vdots\, f_i\\ \Box A_i\end{matrix}}{\Box A_i}\ \mathcal{E}+ \quad \cdots \quad \begin{matrix}\Gamma_n\\ \vdots\, f_n\\ \Box A_n\end{matrix} \quad \begin{matrix}[A_1,\cdots,A_n],\Delta\\ \vdots\, g\\ B\end{matrix}}{\Box B}\ \Box\mathcal{I} \qquad \rightsquigarrow_{\pi\Box}$$

$\rightsquigarrow_{\pi\Box}$

$$\dfrac{\begin{matrix}\Gamma_1\\ \vdots\, f_1\\ \Box A_1\end{matrix} \quad \cdots \quad \dfrac{\begin{matrix}\vdots\, t\end{matrix} \quad \begin{matrix}\Gamma_i\\ \vdots\, f_i\\ \Box A_i\end{matrix}}{\Box A_i}\,\mathcal{E}+ \quad \cdots \quad \begin{matrix}\Gamma_n\\ \vdots\, f_n\\ \Box A_n\end{matrix} \quad \begin{matrix}[A_1,\cdots,A_n],\Delta\\ \vdots\, g\\ B\end{matrix}}{\Box B}\,\Box\mathcal{J} \qquad \rightsquigarrow_{\pi\Box}$$

$$\rightsquigarrow_{\pi\Box} \qquad \dfrac{\begin{matrix}\vdots\, t\end{matrix} \quad \dfrac{\begin{matrix}\Gamma_1\\ \vdots\, f_1\\ \Box A_1\end{matrix} \quad \cdots \quad \begin{matrix}\Gamma_i\\ \vdots\, f_i\\ \Box A_i\end{matrix} \quad \cdots \quad \begin{matrix}\Gamma_n\\ \vdots\, f_n\\ \Box A_n\end{matrix} \quad \begin{matrix}[A_1,\cdots,A_n],\Delta\\ \vdots\, g\\ B\end{matrix}}{\Box B}\,\Box\mathcal{J}}{\Box B}\,\mathcal{E}+$$

where $\mathcal{E}+$ is $\vee\mathcal{E}$ or $\perp_J$.

Lemma

Deductions in IEL^{-} *strongly normalise w.r.t. the rewriting system* $\Xi + \rho_{\Box} + \pi_{\Box}$.

Lemma

Deductions in IEL^{-} *strongly normalise w.r.t. the rewriting system* $\Xi + \rho_{\Box} + \pi_{\Box}$.

Proof.

We define a new translation $\langle - \rangle$ from our modal λ-calculus to simple type theory with products, sums, unit and empty types as follows:

$$\begin{aligned} &\cdots \\ &\cdots \\ \langle \Box A \rangle \quad &:= \quad \langle A \rangle \vee q \end{aligned}$$

$$\begin{aligned} &\cdots \\ &\cdots \\ &\langle \mathsf{box}[x_1, \cdots, x_n].\, g \ \mathsf{with}\ f_1, \cdots, f_n \rangle := \\ &:= \mathsf{C}(\langle f_n \rangle, x_n. \cdots \mathsf{C}(\langle f_2 \rangle, x_2.\mathsf{C}(\langle f_1 \rangle, x_1.\mathsf{in}_1(\langle g \rangle), y_1.\mathsf{in}_2(y_1)), y_2.\mathsf{in}_2(y_2)) \cdots, y_n.\mathsf{in}_2(y_n)) \end{aligned}$$

where q is an arbitrary atom.

Lemma

Deductions in IEL^{-} *strongly normalise w.r.t. the rewriting system* $\Xi + \rho_{\Box} + \pi_{\Box}$.

Proof.

$$\dfrac{\begin{array}{c}\Gamma_1\\ \vdots\, f_1\\ \Box A_1\end{array} \quad \begin{array}{c}\Gamma_2\\ \vdots\, f_2\\ \Box A_2\end{array} \quad \begin{array}{c}[A_1, A_2], \Delta\\ \vdots\, g\\ B\end{array}}{\Box B}\,\Box\mathcal{I} \quad \overset{\langle\,\rangle}{\longmapsto}$$

$$\overset{\langle\,\rangle}{\longmapsto} \quad \dfrac{\begin{array}{c}\langle\Gamma_2\rangle\\ \vdots\,\langle f_2\rangle\\ \langle A_2\rangle \vee q\end{array} \quad \dfrac{\begin{array}{c}\langle\Gamma_1\rangle\\ \vdots\,\langle f_1\rangle\\ \langle A_1\rangle \vee q\end{array} \quad \dfrac{\begin{array}{c}[\langle A_1\rangle]^1, [\langle A_2\rangle]^2, \langle\Delta\rangle\\ \vdots\,\langle g\rangle\\ \langle B\rangle\end{array}}{\langle B\rangle \vee q}\,\vee\mathcal{I}_1 \quad \dfrac{[q]^1}{\langle B\rangle \vee q}\,\vee\mathcal{I}_2}{\langle B\rangle \vee q}\,\vee\mathcal{E}{:}1 \quad \dfrac{[q]^2}{\langle B\rangle \vee q}\,\vee\mathcal{I}_2}{\langle B\rangle \vee q}\,\vee\mathcal{E}{:}2$$

It is now straightforward to check that both $\rho_{\Box}$ *and* $\pi_{\Box}$ are preserved by this translation. ⊠

Lemma

Deductions in IEL^{-} *strongly normalise w.r.t. the rewriting system* $\Xi + \rho_{\Box} + \pi_{\Box}$.

Corollary

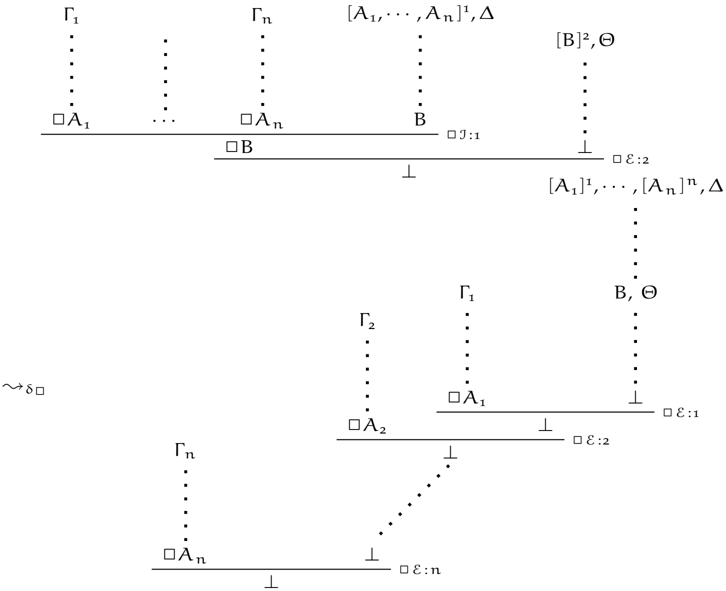
Every IEL^{-}*-deduction uniquely reduces to a normal* IEL^{-}*-deduction w.r.t.* $\Xi + \rho_{\Box} + \pi_{\Box}$.

$$\dfrac{\dfrac{\begin{matrix}\Gamma_1\\ \vdots\\ \Box A_1\end{matrix} \quad \begin{matrix}\vdots\\ \dots\end{matrix} \quad \begin{matrix}\Gamma_n\\ \vdots\\ \Box A_n\end{matrix} \quad \begin{matrix}[A_1,\cdots,A_n]^1,\Delta\\ \vdots\\ B\end{matrix}}{\Box B}\,\Box\mathcal{I}{:}1 \qquad \begin{matrix}[B]^2,\Theta\\ \vdots\\ \bot\end{matrix}}{\bot}\,\Box\mathcal{E}{:}2 \qquad \rightsquigarrow_{\delta\Box}$$

$\rightsquigarrow_{\delta\Box}$

Detours for Knowledge

$\mathcal{I}$-$\mathcal{E}$ Cases



Detours for Knowledge

$\mathcal{E}$-$\mathcal{I}$ Cases

$$\dfrac{\cdots \qquad \dfrac{\begin{array}{c}\Gamma \\ \vdots \\ \Box\bot\end{array} \qquad [\bot]}{\bot}\,\Box\mathcal{E}}{\Box\bot}\,\Box\mathcal{I} \qquad \rightsquigarrow_{\gamma\Box}$$

Detours for Knowledge

$\mathcal{E}$-$\mathcal{J}$ Cases

$$\dfrac{\cdots \qquad \dfrac{\begin{array}{c}\Gamma\\ \vdots\\ \Box\bot\end{array} \qquad [\bot]}{\bot}\,\Box\mathcal{E}}{\Box\bot}\,\Box\mathcal{J} \quad \rightsquigarrow_{\gamma\Box} \quad \cdots \quad \begin{array}{c}\Gamma\\ \vdots\\ \Box\bot\end{array}$$

Theorem

Deductions in IEL *strongly normalise w.r.t. the rewriting system* $\Xi + \rho_{\Box} + \pi_{\Box} + \delta_{\Box} + \gamma_{\Box}$.

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Deductions in IEL *strongly normalise w.r.t. the rewriting system* $\Xi + \rho_\Box + \pi_\Box + \delta_\Box + \gamma_\Box$.

Proof.

Tweak the translation $\langle - \rangle$ from our modal λ-calculus for IEL^- to simple type theory with products, sums, unit and empty types as follows:

$$\begin{aligned} &\dots \\ &\dots \\ \langle \Box A \rangle \quad &:= \quad \langle A \rangle \vee \bot \end{aligned}$$

$\dots$
$\dots$
$\langle \mathsf{box}[x_1, \cdots, x_n].\, \mathsf{g}\ \mathsf{with}\ \mathsf{f}_1, \cdots, \mathsf{f}_n \rangle :=$
$:= \mathsf{C}(\langle \mathsf{f}_n \rangle, x_n. \cdots \mathsf{C}(\langle \mathsf{f}_2 \rangle, x_2.\mathsf{C}(\langle \mathsf{f}_1 \rangle, x_1.\mathsf{in}_1(\langle \mathsf{g} \rangle), y_1.\mathsf{in}_2(y_1)), y_2.\mathsf{in}_2(y_2)) \cdots, y_n.\mathsf{in}_2(y_n))$
$\langle \mathsf{unbox}\ \mathsf{f}\ \mathsf{with}\ x.\mathsf{g} \rangle := \mathsf{C}(\langle \mathsf{f} \rangle, x.\langle \mathsf{g} \rangle, y.y).$

Theorem

Deductions in IEL *strongly normalise w.r.t. the rewriting system* $\Xi + \rho_\Box + \pi_\Box + \delta_\Box + \gamma_\Box$.

Proof.

$$\dfrac{\begin{matrix}\Gamma \\ \vdots\, f \\ \Box A\end{matrix} \qquad \begin{matrix}[A], \Delta \\ \vdots\, g \\ \bot\end{matrix}}{\bot}\ \Box\mathcal{E} \quad \overset{\langle\,\rangle}{\longmapsto} \quad \dfrac{\begin{matrix}\langle\Gamma\rangle \\ \vdots\, \langle f\rangle \\ \langle A\rangle \vee \bot\end{matrix} \qquad \begin{matrix}[\langle A\rangle]^1, \langle\Delta\rangle \\ \vdots\, \langle g\rangle \\ \bot\end{matrix} \qquad [\bot]^1}{\bot}\ \vee\mathcal{E}{:}1$$

This translation preserves $\delta_\Box$ and $\gamma_\Box$ too, by means of applications of rewritings in Ξ concerning $\vee$-detours and permutations. ☒

Theorem (Subformula Property)

Every formula B *occurring in a normal* L*-deduction* f *of* A *from assumptions* Γ *is a subformula of* A *or of some formula in* Γ.

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Every formula B *occurring in a normal* L*-deduction* f *of* A *from assumptions* Γ *is a subformula of* A *or of some formula in* Γ.

Corollary

1. ***Consistency:*** *The logics of intuitionistic belief and intutionistic knowledge are consistent.*
2. ***Decidability:*** *The logics of intuitionistic belief and intutionistic knowledge are decidable.*
3. ***Canonicity:*** *In any normal* L*-deduction of* A*, the last rule applied is the introduction rule for the main connective of* A.
4. ***Admissibility of Reflection:*** *If* $\mathsf{L} \vdash \Box A$*, then* $\mathsf{L} \vdash A$.
5. ***Disjunction Property:*** *If* $\mathsf{L} \vdash A \vee B$*, then* $\mathsf{L} \vdash A$ *or* $\mathsf{L} \vdash B$.
6. $\Box$***-Primality:*** *If* $\mathsf{L} \vdash \Box A \vee \Box B$*, then* $\mathsf{L} \vdash A$ *or* $\mathsf{L} \vdash B$.
7. ***Modal Disjunction Property:*** *If* $\mathsf{L} \vdash \Box(A \vee B)$*, then* $\mathsf{L} \vdash \Box A$ *or* $\mathsf{L} \vdash \Box B$.
8. ***Proper Inclusion:*** $\mathsf{IEL}^{-} \subsetneq \mathsf{IEL}$.

Outline

"Setting the Stage"

Prologue
- Origins
- BHK Explanation
- Axiomatisation(s)
- Relational Semantics

Proofs
- Natural Deduction for Intuitionistic Logic
- Natural Deduction for Intuitionistic Epistemic Logics

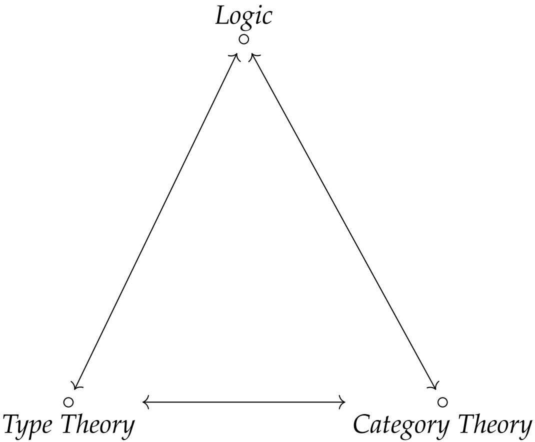
Programs
- Curry-Howard for NJ_p
- Modal Type Theory for Epistemic States
- Normalisation Theorems

Arrows
- Curry-Howard-Lambek for NJ_p
- Categorical Structures for Intuitionistic Epistemic Logics
- Soundness and Completeness

Epilogue

Curry-Howard-Lambek Correspondence

Computational Trinitarianism



Curry-Howard-Lambek Correspondence

A Glossary [LS88]

Logic	Type Theory	Category Theory
proposition	type	object
proof	term	arrow
theorem	inhabitant	element-arrow
conjunction	product type	product
true	unit type	terminal object
implication	function type	exponential
disjunction	sum type	(weak) coproduct
false	empty type	(weak) initial object

Definition

A **category** $\mathscr{C}$ consists of the following data:

- a collection of **objects** $\mathscr{C}_0$, generally denoted $\mathscr{C}$ also;
- a collection of **arrows** $\mathscr{C}_1$;
- to each arrow $f : \mathscr{C}_1$, a pair of objects $\langle \mathsf{source}(f), \mathsf{target}(f) \rangle$ is associated. We write $f : X \to Y$. The collection of arrows from X to Y is denoted by $\mathsf{hom}(X, Y)$;
- to each object $X : \mathscr{C}_0$, an arrow $\mathrm{id}_X : X \to X$ is associated.
- to each pair of arrows f, g such that $\mathsf{source}(g) = \mathsf{target}(f)$ is associated an arrow $g \circ f : \mathsf{source}(f) \to \mathsf{target}(g)$.

These data need to satisfy specific conditions:

Associativity: For any composable arrows f, g and g, h, we have $h \circ (g \circ f) = (h \circ g) \circ f$;

Identity: For any $f : X \to Y$, we have $\mathrm{id}_Y \circ f = f = f \circ \mathrm{id}_X$.

Definition

Let $\mathscr{C}$ and $\mathscr{D}$ be categories. A **functor** $\mathfrak{F} : \mathscr{C} \to \mathscr{D}$ consists of the following data:

- A function $\mathfrak{F}_0 : \mathscr{C} \to \mathscr{D}$, denoted $\mathfrak{F}$ also.
- For each $X, Y : \mathscr{C}$, a function $\mathfrak{F}_{X,Y} : \mathrm{hom}_{\mathscr{C}}(X, Y) \to \mathrm{hom}_{\mathscr{D}}(\mathfrak{F}X, \mathfrak{F}Y)$, denoted $\mathfrak{F}$ also;

satisfying the following conditions:

(i) For each $X : \mathscr{C}$, we have $\mathfrak{F}(\mathrm{id}_X) = 1_{\mathfrak{F}(X)}$; and

(ii) For each $X, Y, Z : \mathscr{C}$, and $f : \mathrm{hom}_{\mathscr{C}}(X, Y)$ and $g : \mathrm{hom}_{\mathscr{C}}(Y, Z)$, we have $\mathfrak{F}(g \circ f) = \mathfrak{F}g \circ \mathfrak{F}f$.

Definition

For functors $\mathfrak{F}, \mathfrak{G} : \mathscr{C} \to \mathscr{D}$, a **natural transformation** $\gamma : \mathfrak{F} \Rightarrow \mathfrak{G}$ consists of the following data:

- For each $X : \mathscr{C}$, an arrow $\gamma_X : \mathsf{hom}_{\mathscr{D}}(\mathfrak{F}X, \mathfrak{G}X)$, giving the **components** of γ;

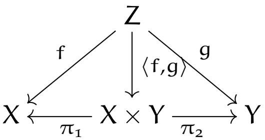
satisfying the **naturality** condition

For each $X, Y : \mathscr{C}$ and $f : \mathsf{hom}_{\mathscr{C}}(X, Y)$, we have $\mathfrak{G}f \circ \gamma_X = \gamma_Y \circ \mathfrak{F}f$.

Category Theory

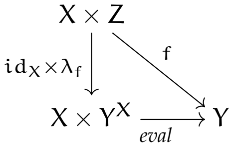
A Primer [Mac13]

- **Product of objects** X, Y**:** Object $X \times Y$ of $\mathscr{C}$ together with projection arrows $\pi_1 : X \times Y \to X$, $\pi_2 : X \times Y \to Y$ such that any diagram



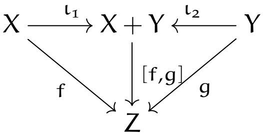
  commutes for a unique $\langle f, g \rangle : Z \to X \times Y$.
- **Terminal object:** Object 1 of $\mathscr{C}$ such that for any other $X : \mathscr{C}$, there exists a unique arrow $!_X : X \to 1$.

- **Exponential of objects Y and X:** An object Y^X of $\mathscr{C}$ together with an arrow *eval* $: X \times Y^X \to Y$ such that any diagram



commutes for a unique $\lambda_f : Z \to Y^X$.

- **Coproduct of objects** X, Y**:** Object $X + Y$ of $\mathscr{C}$ together with injection arrows $\iota_1 : X \to X + Y$, $\iota_2 : Y \to X + Y$ such that any diagram



  commutes for a unique $[f, g] : X + Y \to Z$.
- **Initial object:** Object 0 of $\mathscr{C}$ such that for any other $X : \mathscr{C}$, there exists a unique arrow $0_X : 0 \to X$.

Cartesian closed categories (CCCs) provide the semantics *and* proof theory of disjunction-free propositional intuitionistic logic:

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- Objects correspond to formulae, and an arrow from A to B corresponds to a deduction $A \vdash_{\mathsf{NJ}} B$ between the corresponding formulae;

Cartesian closed categories (CCCs) provide the semantics *and* proof theory of disjunction-free propositional intuitionistic logic:

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- Conjunction is modelled by products, by using the adjunction

$$C \vdash_{\mathsf{NJ}_\mathrm{p}} A \wedge B \qquad \simeq \qquad C \vdash_{\mathsf{NJ}_\mathrm{p}} A \,\&\, C \vdash_{\mathsf{NJ}_\mathrm{p}} B;$$

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$$A \wedge B \vdash_{\mathsf{NJ_p}} C \qquad \simeq \qquad A \vdash_{\mathsf{NJ_p}} B \to C;$$

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- Implication is modelled by exponentials, by using the adjunction
$$A \wedge B \vdash_{\mathsf{NJ_p}} C \qquad \simeq \qquad A \vdash_{\mathsf{NJ_p}} B \to C;$$
- Top is modelled by terminal object 1.

If a CCC has finite (weak) coproducts it is called bi-Cartesian closed.
Bi-CCCs provide the semantics *and* proof theory of propositional intuitionistic logic:

If a CCC has finite (weak) coproducts it is called bi-Cartesian closed.
Bi-CCCs provide the semantics *and* proof theory of propositional intuitionistic logic:

- Disjunction is modelled by (weak) coproducts, by using the adjunction

$$A \vee B \vdash_{\mathsf{NJ}} C \qquad \simeq \qquad A \vdash_{\mathsf{NJ}} C \,\&\, B \vdash_{\mathsf{NJ}} C;$$

- Bottom is modelled by (weak) initial object 0.

Given a CCCat $\mathscr{C}$, a monoidal endofunctor consists of a functor $\mathfrak{F} : \mathscr{C} \to \mathscr{C}$ together with

- a natural transformation

$$m_{A,B} : \mathfrak{F}A \times \mathfrak{F}B \to \mathfrak{F}(A \times B);$$

- a morphism

$$m_1 : 1 \to \mathfrak{F}1,$$

preserving the monoidal structure of $\mathscr{C}$. These are called structure morphisms of $\mathfrak{F}$.

Given any category $\mathscr{C}$, an endofunctor $\mathfrak{F} : \mathscr{C} \to \mathscr{C}$ is pointed iff there exists a natural transformation

$$\pi : \mathrm{Id}_{\mathscr{C}} \Rightarrow \mathfrak{F}$$
$$\pi_A : A \to \mathfrak{F}A$$

$$\begin{array}{ccc} A & \xrightarrow{\pi_A} & \mathfrak{F}A \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mathfrak{F}f} \\ B & \xrightarrow[\pi_B]{} & \mathfrak{F}B \end{array}$$

π is called the point of $\mathfrak{F}$.

Given a monoidal category $\mathscr{C}$, and monoidal endofunctors $\mathfrak{F}, \mathfrak{G} : \mathscr{C} \to \mathscr{C}$, a natural transformation $\kappa : \mathfrak{F} \Rightarrow \mathfrak{G}$ is monoidal when the following commute:

$$\begin{array}{ccc}
\mathfrak{F}A \times \mathfrak{F}B & \xrightarrow{m^{\mathfrak{F}}_{A,B}} & \mathfrak{F}(A \times B) \\
{\scriptstyle \kappa_A \times \kappa_B}\downarrow & & \downarrow{\scriptstyle \kappa_{A \times B}} \\
\mathfrak{G}A \times \mathfrak{G}B & \xrightarrow[m^{\mathfrak{G}}_{A,B}]{} & \mathfrak{G}(A \times B)
\end{array}$$

and

$$\begin{array}{ccc}
1 & \xrightarrow{m^{\mathfrak{F}}_{1}} & \mathfrak{F}(1) \\
\| & & \downarrow{\scriptstyle \kappa_1} \\
1 & \xrightarrow[m^{\mathfrak{G}}_{1}]{} & \mathfrak{G}(1)
\end{array}$$

Definition

An IEL^--category is given by a bi-CCCat $\mathscr{C}$ together with a monoidal pointed endofunctor $\mathfrak{K}$ whose point κ is monoidal.
When $\mathfrak{K}$ preserves the initial object 0 of $\mathscr{C}$ up-to-isomorphism, we say that $\mathfrak{K}$ is *dense*, and $\mathscr{C}$ is an IEL-category.

$$\dfrac{\begin{matrix}\Gamma \\ \vdots \\ \Box A\end{matrix} \qquad [A]}{\Box A}\ \Box\mathcal{I} \qquad \rightsquigarrow_{\eta_\Box}$$

$$\dfrac{\begin{matrix}\Gamma \\ \vdots \\ \Box A\end{matrix} \qquad [A]}{\Box A}\,\Box\mathcal{I} \quad \rightsquigarrow_{\eta_\Box} \quad \begin{matrix}\Gamma \\ \vdots \\ \Box A\end{matrix}$$

Lemma

L*-deductions strongly normalise w.r.t. the rewriting system* $\Xi + \rho_\Box + \delta_\Box + \gamma_\Box + \eta_\Box$

Theorem

Let $\mathscr{C}$ be an IEL^--category. Then there is a canonical interpretation $[\![-]\!]$ of IEL^- in $\mathscr{C}$ such that

- *a formula A is mapped to a $\mathscr{C}$-object $[\![A]\!]$;*
- *a deduction f of $A_1, \cdots, A_n \vdash_{\mathsf{IEL}^-} B$ is mapped to an arrow*

$$[\![f]\!] : [\![A_1]\!] \times \cdots \times [\![A_n]\!] \to [\![B]\!];$$

- *for any two deductions f and g which are equal modulo $\Xi + \rho_\Box + \eta_\Box$-rewritings, we have $[\![f]\!] = [\![g]\!]$.*

The analogous soundness result holds for any IEL-category, by considering the system of rewritings $\Xi + \rho_\Box + \delta_\Box + \gamma_\Box$ extended by $\eta_\Box$.

Proof Sketch.

By structural induction on $f : \vec{A} \vdash_{\mathsf{IEL}^-} B$.
The intuitionistic cases are interpreted according to the remarks about bi-CCCats.

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Let $\langle \Box, m, \kappa \rangle$ be the monoidal pointed endofunctor of $\mathscr{C}$, its structure morphisms, and its point.
The deduction $\dfrac{f_1 : \Gamma_1 \vdash \Box A_1 \quad \cdots \quad f_n : \Gamma_n \vdash \Box A_n \quad g : [A_1, \cdots, A_n], C_1, \cdots, C_m \vdash B}{\Gamma_1, \cdots, \Gamma_n, C_1, \cdots, C_m \vdash \Box B}$ is mapped to

$$(\Box[\![g]\!]) \circ m_{[\![A_1]\!],\cdots,[\![A_n]\!],[\![C_1]\!],\cdots,[\![C_m]\!]} \circ [\![f_1]\!] \times \cdots \times [\![f_n]\!] \times \kappa_{[\![C_1]\!]} \times \cdots \times \kappa_{[\![C_m]\!]}.$$

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where $m_{X_1, \cdots, X_n}$ is defined inductively as

$$m_{X_1, \cdots, X_{n-1}, X_n} := m_{X_1 \times \cdots \times X_{n-1}, X_n} \circ (m_{X_1, \cdots, X_{n-1}}) \times id_{\Box X_n}.$$

Identity modulo $\eta_\Box$ holds in the category $\mathscr{C}$ by functoriality of $\Box$.

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The intuitionistic cases are interpreted according to the remarks about bi-CCCats.
Let $\langle \Box, m, \kappa \rangle$ be the monoidal pointed endofunctor of $\mathscr{C}$, its structure morphisms, and its point.
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$$(\Box[\![g]\!]) \circ m_{[\![A_1]\!],\cdots,[\![A_n]\!],[\![C_1]\!],\cdots,[\![C_m]\!]} \circ [\![f_1]\!] \times \cdots \times [\![f_n]\!] \times \kappa_{[\![C_1]\!]} \times \cdots \times \kappa_{[\![C_m]\!]}.$$

where $m_{X_1,\cdots,X_n}$ is defined inductively as

$$m_{X_1,\cdots,X_{n-1},X_n} := m_{X_1 \times \cdots \times X_{n-1},X_n} \circ (m_{X_1,\cdots,X_{n-1}}) \times id_{\Box X_n}.$$

Identity modulo $\eta_\Box$ holds in the category $\mathscr{C}$ by functoriality of $\Box$.
Identity modulo $\rho_\Box$ is also valid by naturality of m and κ.

Proof Sketch.

By structural induction on $f : \vec{A} \vdash_{\mathsf{IEL}^-} B$.
The intuitionistic cases are interpreted according to the remarks about bi-CCCats.
Let $\langle \Box, m, \kappa \rangle$ be the monoidal pointed endofunctor of $\mathscr{C}$, its structure morphisms, and its point.
The deduction $\dfrac{f_1 : \Gamma_1 \vdash \Box A_1 \quad \cdots \quad f_n : \Gamma_n \vdash \Box A_n \quad g : [A_1, \cdots, A_n], C_1, \cdots, C_m \vdash B}{\Gamma_1, \cdots, \Gamma_n, C_1, \cdots, C_m \vdash \Box B}$ is mapped to

$$(\Box[\![g]\!]) \circ m_{[\![A_1]\!],\cdots,[\![A_n]\!],[\![C_1]\!],\cdots,[\![C_m]\!]} \circ [\![f_1]\!] \times \cdots \times [\![f_n]\!] \times \kappa_{[\![C_1]\!]} \times \cdots \times \kappa_{[\![C_m]\!]}.$$

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Identity modulo $\eta_\Box$ holds in the category $\mathscr{C}$ by functoriality of $\Box$.
Identity modulo $\rho_\Box$ is also valid by naturality of m and κ.
For IEL, we furthermore have the density property of the functor $\Box$, which validates identity of deductions modulo $\delta_\Box$ and $\gamma_\Box$. ⊠

Theorem

If the interpretation of two L*-deductions is equal in all* L*-categories, then the two deductions are equal modulo* $\Xi + \rho_\Box + \delta_\Box + \gamma_\Box + \eta_\Box$.

Theorem

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Proof Sketch.

We proceed by constructing a term model for the modal λ-calculus for L-deductions. Consider the following category $\mathcal{M}$:

- its objects are formulas;
- an arrow $f : A \to B$ is an L-deduction of B from A modulo rewritings in $\Xi + \rho_\Box + \delta_\Box + \gamma_\Box + \eta_\Box$;
- identities are given by assuming a hypothesis;
- composition is given by transitivity of deductions.

$\mathcal{M}$ has the structure required to be an L-category in virtue of the rewriting system imposed on deductions, so that if an equation between interpreted L-deductions holds in all L-categories, then it holds also in $\mathcal{M}$, so that those deductions are equal whenever they normalise to the same proof w.r.t. $\Xi + \rho_\Box + \delta_\Box + \gamma_\Box + \eta_\Box$-rewritings. ⊠

Outline

"Setting the Stage"

Prologue
- Origins
- BHK Explanation
- Axiomatisation(s)
- Relational Semantics

Proofs
- Natural Deduction for Intuitionistic Logic
- Natural Deduction for Intuitionistic Epistemic Logics

Programs
- Curry-Howard for NJ_p
- Modal Type Theory for Epistemic States
- Normalisation Theorems

Arrows
- Curry-Howard-Lambek for NJ_p
- Categorical Structures for Intuitionistic Epistemic Logics
- Soundness and Completeness

Epilogue

Recap

- We developed natural deduction calculi for intuitionistic epistemic logics, situating them within the Curry-Howard-Lambek correspondence.
- We established their proof-theoretic well-behaviour by proving key structural properties, including normalization theorems.
- We provided a categorical interpretation of epistemic modalities as functors, giving a precise proof-theoretic semantics to intuitionistic knowledge and belief.

Conclusions

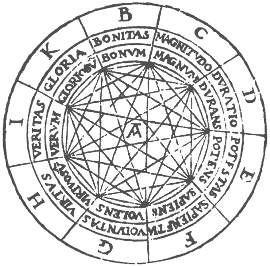
[PB21, PB25]

Recap

- ▷ We developed natural deduction calculi for intuitionistic epistemic logics, situating them within the Curry-Howard-Lambek correspondence.
- ▷ We established their proof-theoretic well-behaviour by proving key structural properties, including normalization theorems.
- ▷ We provided a categorical interpretation of epistemic modalities as functors, giving a precise proof-theoretic semantics to intuitionistic knowledge and belief.

Future Directions

- A comparative analysis and inter-translatability results with existing sequent calculi for these logics.
- Modular extensions of our base calculus (IEL^{-}) to model richer logics for/than epistemic systems.
- Deeper investigation into the computational interpretations of our systems w.r.t. functional programming paradigms.



Many Thanks for Your Attention!

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