Unified Correspondence as a Proof-Theoretic Tool

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Motivation

Main question: which axioms give rise to analytic rules?

[Kracht 1996], [Negri 2005], [Ciabattoni et al. 2008], [Ciabattoni et al. 2009], [Goré et al. 2011], [Ciabattoni et al. 2012], [Lellmann et al. 2013]...

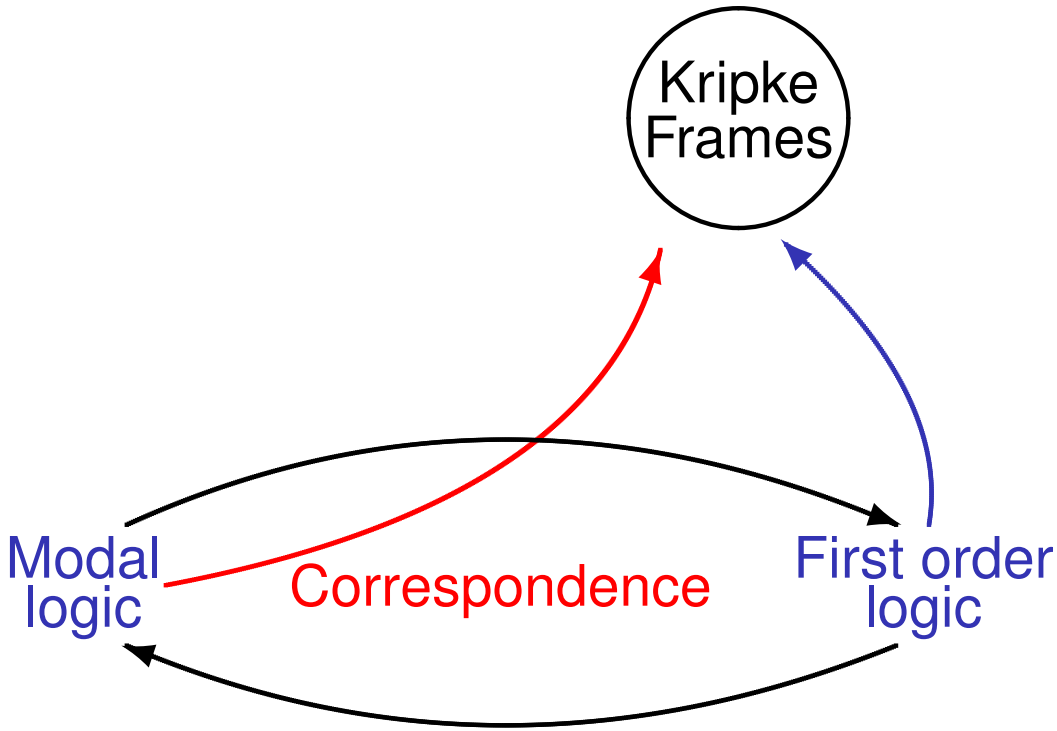
Correspondence theory can help in answering this question!

- Formal connections between **correspondence theory** and **display calculi**.

- **Primitive formulas** [Kracht '96] for classical modal logic **K** generalised to **primitive inequalities** for general **LE-logics**.

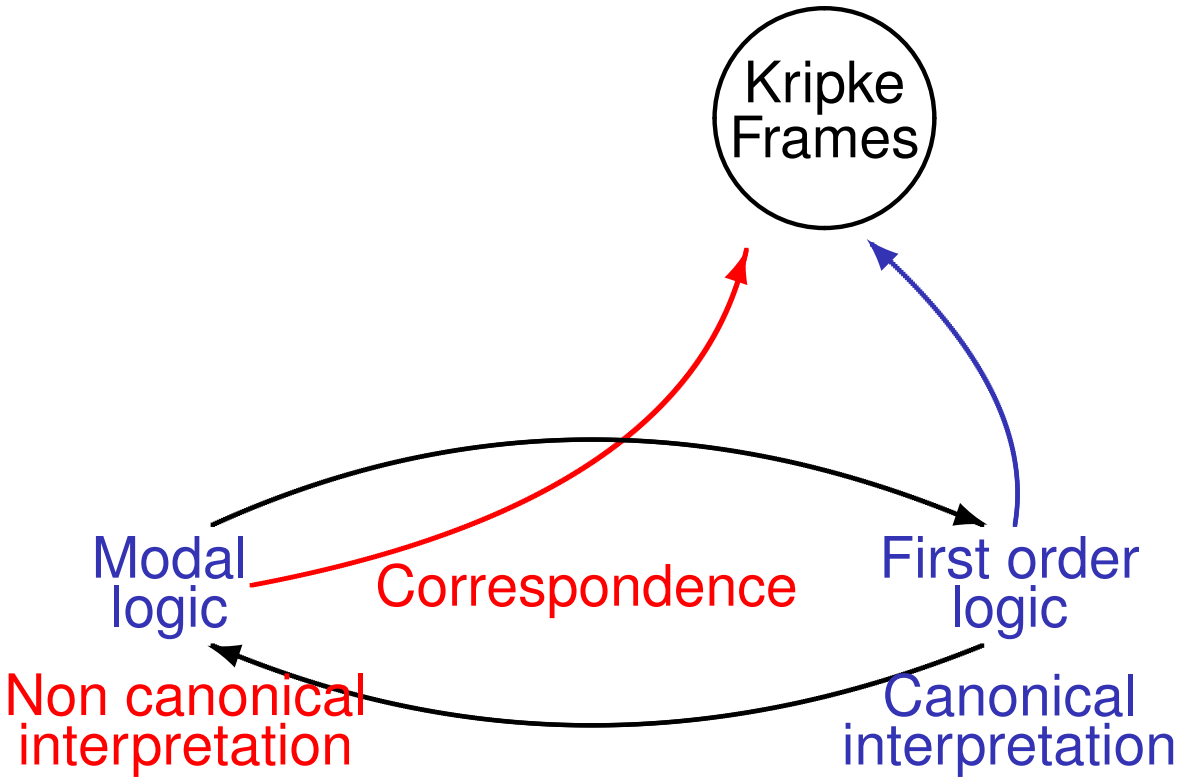
Correspondence via Duality

Correspondence theory arises semantically:



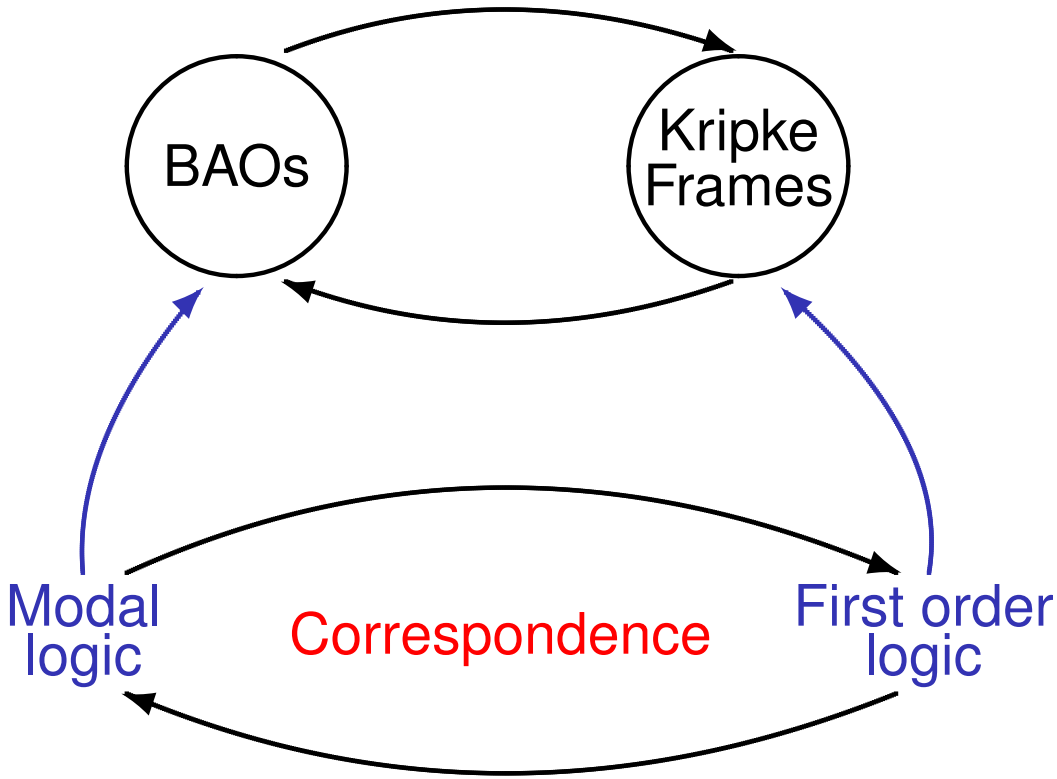
Correspondence via Duality

An asymmetry:



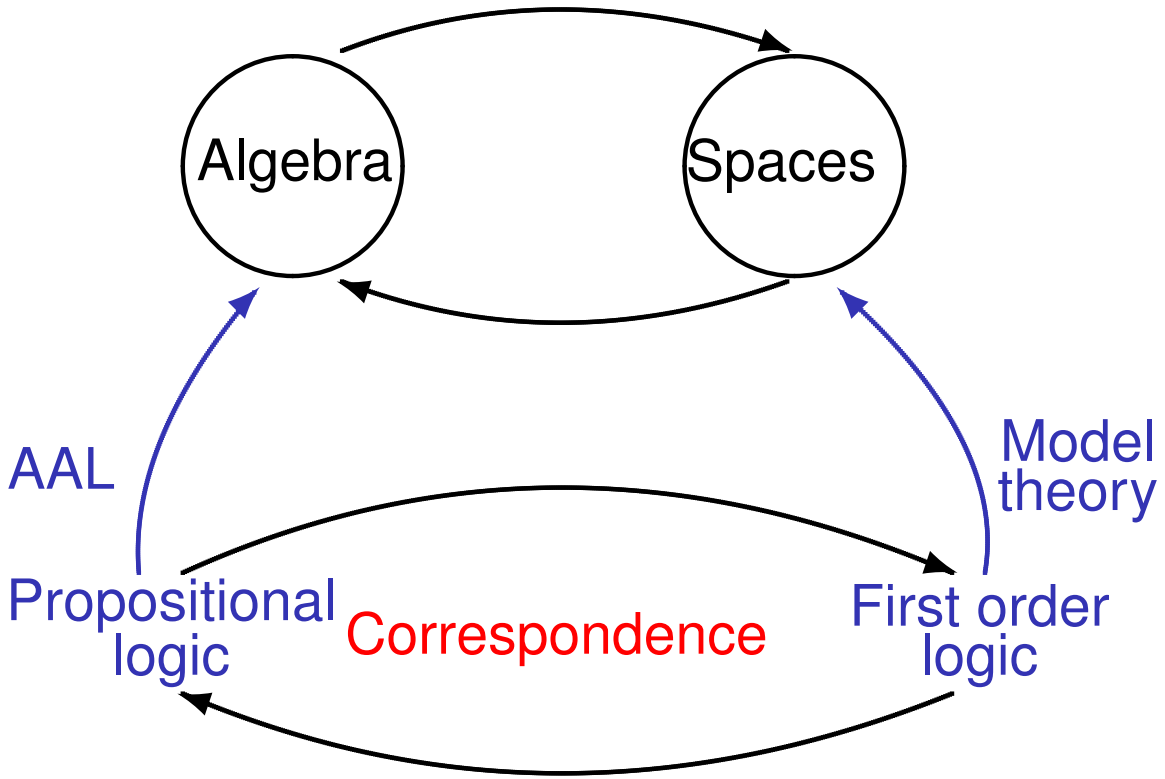
Correspondence via Duality

Symmetry re-established via **duality**:



Correspondence via Duality

Correspondence available not just for modal logic:



- specific correspondences as logical reflections of dualities
- dual characterizations as instances of **unified correspondence**

Unified correspondence

Hybrid logics
[CR15]

LE-logics
[CP12, CP19, CPS]

Constructive, slanted canonicity
[CP20, DP]

Mu-calculi
[CFPS15, CGP14]

Proper display calculi
[GMPTZ]

Regular DLE-logics
Kripke frames with impossible worlds
[PSZ15a]

Jónsson-style vs Sambin-style canonicity
[PSZ15b]

Finite lattices and monotone ML
[FPS15]

Canonicity via pseudo-correspondence
[CPSZ]

Proper Display Calculi

Natural generalization of sequent calculi.
Sequents $X \vdash Y$, where X, Y are structures:
A, $A; B$, ... $X > Y$, ...

structural symbols assemble and disassemble structures
operational symbols assemble formulas.

Main feature: display property

$$\dfrac{\dfrac{Y \vdash X > Z}{X; Y \vdash Z}}{\dfrac{Y; X \vdash Z}{X \vdash Y > Z}}$$

display property: **adjunction/residuation** at the structural level.
Canonical proof of cut elimination

Proper Display Calculi

Theorem (Canonical cut elimination)

If a calculus satisfies the properties below, then it enjoys cut elimination.

- C1: structures can disappear, formulas are **forever**;
- **tree-traceable** formula-occurrences, via suitably defined congruence:
  - C2: same shape, C3: non-proliferation, C4: same position;
- C5: **principal = displayed**;
- C6, C7: rules are closed under **uniform substitution** of congruent parameters;
- C8: **reduction strategy** exists when cut formulas are both principal.

(D)LE-languages and expansions

$$\varphi ::= p \mid \bot \mid \top \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid f(\overline{\varphi}) \mid g(\overline{\varphi})$$

where $p \in \mathrm{PROP}$, $f \in \mathcal{F}$, $g \in \mathcal{G}$.

Str.	I		;		>		H		K	
Op.	$\top$	$\bot$	$\wedge$	$\vee$	$(\succ\!\!-)$	$(\rightarrow)$	f			g

Str.	H_i		K_h	
Op.		$(f_i^\sharp)$	$(g_h^\flat)$	

  for $\varepsilon_f(i) = \varepsilon_g(h) = 1$

Str.	H_i		K_h	
Op.	$(f_i^\sharp)$			$(g_h^\flat)$

  for $\varepsilon_f(i) = \varepsilon_g(h) = \partial$

Introduction rules for $f \in \mathcal{F}$ and $g \in \mathcal{G}$

$$f_L \ \frac{H(A_1, \ldots, A_{n_f}) \vdash X}{f(A_1, \ldots, A_{n_f}) \vdash X} \qquad \frac{X \vdash K(A_1, \ldots, A_{n_g})}{X \vdash g(A_1, \ldots, A_{n_g})} \ g_R$$

$$f_R \ \frac{\left(X_i \vdash A_i \quad A_j \vdash X_j \quad | \quad \varepsilon_f(i) = 1 \quad \varepsilon_f(j) = \partial\right)}{H(X_1, \ldots, X_{n_f}) \vdash f(A_1, \ldots, A_{n_f})}$$

$$g_L \ \frac{\left(A_i \vdash X_i \quad X_j \vdash A_j \quad | \quad \varepsilon_g(i) = 1 \quad \varepsilon_g(j) = \partial\right)}{g(A_1, \ldots, A_{n_g}) \vdash K(X_1, \ldots, X_{n_g})}$$

Display postulates for $f \in \mathcal{F}$ and $g \in \mathcal{G}$

- If $\varepsilon_f(i) = \varepsilon_g(h) = 1$

$$\frac{H(X_1, \ldots, X_i, \ldots, X_{n_f}) \vdash Y}{X_i \vdash H_i(X_1, \ldots, Y, \ldots, X_{n_f})} \qquad \frac{Y \vdash K(X_1 \ldots, X_h, \ldots X_{n_g})}{K_h(X_1, \ldots, Y, \ldots, X_{n_g}) \vdash X_h}$$

- If $\varepsilon_f(i) = \varepsilon_g(h) = \partial$

$$\frac{H(X_1, \ldots, X_i, \ldots, X_{n_f}) \vdash Y}{H_i(X_1, \ldots, Y, \ldots, X_{n_f}) \vdash X_i} \qquad \frac{Y \vdash K(X_1, \ldots, X_h, \ldots, X_{n_g})}{X_h \vdash K_h(X_1, \ldots, Y, \ldots, X_{n_g})}$$

Algorithmic correspondence for LE

Ackermann **L**emma **B**ased **A**lgorithm

- engined by the Ackermann lemma.
- Reduction rules leading to the Ackermann elimination step.
- Residuation and approximation rules.
- Soundness on **perfect LEs**:
  - approximation: both $\bigvee$-generated by the c. $\vee$-primes and $\bigwedge$-generated by the c. $\wedge$-primes;
  - residuation: all the operations are either right or left adjoints or residuals.

Perfect LEs: the natural semantic environment both for ALBA and for display calculi for LE-logics.

Analytic inductive axioms

For any $\varepsilon : \mathrm{At} \to \{1, \partial\}$ and strict order Ω on At, *positive (γ) and negative (χ) ε^∂-uniform PIA terms*:

$$\begin{array}{rcl} \gamma & := & q \mid \bot \mid \top \mid \gamma \vee \gamma \mid \gamma \wedge \gamma \mid g(\vec{\gamma}/\vec{x}, \vec{\chi}/\vec{y}), \\ \chi & := & p \mid \top \mid \bot \mid \chi \wedge \chi \mid f(\vec{\chi}/\vec{x}, \vec{\gamma}/\vec{y}). \end{array}$$

$\varepsilon(p) = 1$, $\varepsilon(q) = \partial$. *Positive ($\varphi$) and negative ($\psi$) PIA terms* which are not ε^∂-uniform:

$$\begin{array}{rcl} \varphi & := & p \mid \top \wedge \varphi \mid \bot \wedge \varphi \mid \varphi \wedge \varphi \mid \gamma \vee \varphi \mid \chi \to \varphi \mid \psi \to \gamma, \\ \psi & := & q \mid \bot \vee \psi \mid \top \wedge \psi \mid \psi \vee \psi \mid \chi \wedge \psi. \end{array}$$

CVar : PosPIA $\cup$ NegPIA $\to \wp(\mathrm{Var})$; all var's in CVar(φ) (resp. CVar(ψ)) are common upper bounds of var's in γ and χ w.r.t. Ω.

Analytic inductive axioms

$s \vdash t$ such that:

$$\begin{array}{rcl} s & := & \gamma \mid \varphi \mid s \vee s \mid s \wedge s, \\ t & := & \chi \mid \psi \mid t \wedge t \mid t \vee t \mid s \to t. \end{array}$$

Examples:

$$\top \vdash (p \to q) \vee (q \to p)$$

analytic inductive for $\varepsilon(p) = 1$, $\varepsilon(q) = \partial$ and $\Omega = \varnothing$.

$$p \to (q \to r) \vdash (p \to q) \to (p \to r)$$

analytic inductive for $\varepsilon(p) = 1$ and $\varepsilon(q) = \varepsilon(r) = \partial$ and $\Omega = \{(p, q), (r, q)\}$.

$$(p \wedge q) \to r \vdash (p \to r) \vee (q \to r)$$

analytic inductive for $\varepsilon(p) = \varepsilon(q) = \varepsilon(r) = 1$ and $\Omega = \{(p, r), (q, r)\}$.

ALBA-computation of analytic structural rules 1/3

$$
\begin{array}{ll}
 & \forall p \forall q (\top \vdash (p \to q) \vee (q \to p)) \\
\text{iff} & \forall q \forall x \forall y \forall w \forall z ((x \vdash p \;\&\; w \vdash q \;\&\; q \vdash z \;\&\; p \vdash y) \Rightarrow \\
 & \top \vdash (x \to z) \vee (w \to y)) \\
\text{iff} & \forall x \forall y \forall w \forall z ((x \vdash y \;\&\; w \vdash z) \Rightarrow \top \vdash (x \to z) \vee (w \to y)).
\end{array}
$$

The last line in the chain above can be read off as the following analytic structural rule:

$$
\frac{X \vdash Y \qquad W \vdash Z}{\vdash (X \mathrel{\check{\to}} Z) \mathrel{\check{\vee}} (W \mathrel{\check{\to}} Y)}
$$

ALBA-computation of analytic structural rules 2/3

$$
\begin{array}{ll}
 & \forall p \forall q \forall r (p \to (q \to r) \vdash (p \to q) \to (p \to r)) \\
\text{iff} & \forall p q r x y w z ((x \vdash p \to (q \to r) \;\&\; y \vdash p \to q \;\&\; z \vdash p \;\&\; r \vdash w) \Rightarrow \\
 & x \vdash y \to (z \to w)) \\
\text{iff} & \forall p q r x y w z ((q \vdash (p \wedge x) \to r \;\&\; y \vdash p \to q \;\&\; z \vdash p \;\&\; r \vdash w) \Rightarrow \\
 & x \vdash y \to (z \to w)) \\
\text{iff} & \forall x \forall y \forall w \forall z (y \vdash z \to ((z \wedge x) \to w) \Rightarrow x \vdash y \to (z \to w)) \\
\text{iff} & \forall x \forall y \forall w \forall z ((z \wedge y) \wedge (z \wedge x) \vdash w \Rightarrow z \wedge (y \wedge x) \vdash w)
\end{array}
$$

$$
\frac{(Z \mathbin{\hat{\wedge}} Y) \mathbin{\hat{\wedge}} (Z \mathbin{\hat{\wedge}} X) \vdash W}{Z \mathbin{\hat{\wedge}} (Y \mathbin{\hat{\wedge}} X) \vdash W}
$$

As expected, this rule is derivable in the display calculus for intuitionistic propositional logic, given that its corresponding axiom is an intuitionistic tautology.

ALBA-computation of analytic structural rules 3/3

$$
\begin{array}{ll}
 & \forall p \forall q \forall r((p \wedge q) \to r \vdash (p \to r) \vee (q \to r)) \\
\text{iff} & \forall pqrxywzu((x \vdash (p \wedge q) \to r \;\&\; y \vdash p \;\&\; r \vdash z \;\&\; w \vdash q \;\&\; r \vdash u) \Rightarrow \\
 & x \vdash (y \to z) \vee (w \to u)) \\
\text{iff} & \forall rxywzu(((y \wedge w) \wedge x \vdash r \;\&\; r \vdash z \;\&\; r \vdash u) \Rightarrow \\
 & x \vdash (y \to z) \vee (w \to u)) \\
\text{iff} & \forall xywzu(((y \wedge w) \wedge x \vdash z \;\&\; (y \wedge w) \wedge x \vdash u) \Rightarrow \\
 & x \vdash (y \to z) \vee (w \to u)).
\end{array}
$$

The last line in the chain above can be read off as the following analytic structural rule:

$$
\frac{(Y \mathbin{\hat{\wedge}} W) \mathbin{\hat{\wedge}} X \vdash Z \qquad (Y \mathbin{\hat{\wedge}} W) \mathbin{\hat{\wedge}} X \vdash U}{X \vdash (Y \mathbin{\check{\to}} Z) \mathbin{\check{\vee}} (W \mathbin{\check{\to}} U)}
$$

The big five

Conservativity
Soundness
Subformula property
Syntactic completeness [CGPT]
Cut elimination

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