Lévy–Montague reflection is Π^1_1-conservative over WKL_0

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Lévy–Montague reflection in ZF

The cumulative hierarchy:

$$V_0 = \varnothing, \qquad V_{\alpha+1} = \mathcal{P}(V_\alpha), \qquad V_\lambda = \bigcup_{\alpha<\lambda} V_\alpha, \qquad V = \bigcup_{\alpha\in \mathrm{Ord}} V_\alpha.$$

Reflection (Lévy 1960, Montague 1961)

For each formula $\varphi(\vec{x})$, ZF proves that for any $\vec{x}$ if $\varphi(\vec{x})$ then there are arbitrarily large α such that $\vec{x} \in V_\alpha$ and $\varphi^{V_\alpha}(\vec{x})$.

Absoluteness variant

For each formula $\varphi(\vec{x})$, ZF proves that there are arbitrarily large α such that

$$\forall \vec{x} \in V_\alpha \left(\varphi(\vec{x}) \leftrightarrow \varphi^{V_\alpha}(\vec{x})\right).$$

This talk focuses on the absoluteness variant.

Uses and significance of reflection

- *Countable transitive models*: for each finite $\Phi \subseteq \mathrm{ZFC}$, reflection + Löwenheim–Skolem + Mostowski collapse give a countable transitive $(M, \in) \models \Phi$ — the standard starting point of forcing and relative-consistency arguments.
- *Inexhaustibility, made precise* (Montague): ZF proves $\mathrm{Con}(\Phi)$ for each finite $\Phi \subseteq \mathrm{ZF}$; hence no consistent extension of ZF is finitely axiomatizable.
- *Stronger reflection = large cardinals*: for formulas beyond first order, reflection becomes an axiom: V_κ reflects all Π^1_1 sentences iff κ is weakly compact (Hanf–Scott).
- *Conservative foundations* (Feferman 1969): ZFC/S, a *conservative* extension of ZFC by a constant S with $\forall \vec{x} \in S\,(\varphi^S(\vec{x}) \leftrightarrow \varphi(\vec{x}))$ for each φ, supports universe-style category theory without large cardinals.

Systems of Second-Order Arithmetic

Second-order arithmetic speaks about two kinds of objects: *natural numbers* $x, y, \ldots$ and *sets of naturals* $X, Y, \ldots$; the intended model is $(\mathbb{N},\ \mathcal{P}(\mathbb{N}),\ 0, 1, +, \times, <, \in)$.

Reals, complete separable metric spaces, countable algebraic structures, Borel sets, ... can be coded by sets of naturals $\Rightarrow$ a large part of mathematics formalizes here.

The full system Z_2 ("analysis")

Basic arithmetic + induction + *comprehension*: for every formula φ,

$$\exists X\, \forall x\, \big(x \in X \leftrightarrow \varphi(x)\big) \qquad \text{"every definable set exists".}$$

Subsystems allow only some φ's in the comprehension scheme:

- RCA_0: computable (Δ^0_1) comprehension — *the world of computable mathematics*;
- ACA_0: arithmetical comprehension (φ has only number quantifiers) — *the world closed under the Turing jump; same arithmetic strength as Peano arithmetic*.

Lévy–Montague reflection in Z_2

An ω-*model*: a model with the standard naturals, i.e. $(\mathbb{N}, \mathcal{S})$ with $\mathcal{S} \subseteq \mathcal{P}(\mathbb{N})$ — the role of the V_α's.

A *coded* ω-model: a single set Y coding the family of its sections,

$$\mathcal{S} = \{(Y)_y : y \in \mathbb{N}\}, \qquad (Y)_y = \{x : \langle x, y\rangle \in Y\}.$$

$X \dot{\in} Y$: "X is a section of Y"; φ^Y: set quantifiers of φ restricted to sections of Y.

Definition

The reflection scheme Rfn consists of the axioms, for each formula $\varphi(\vec{P}, \vec{x}, \vec{X})$:

$$\forall \vec{P}\, \exists Y \left(\vec{P} \dot{\in} Y \ \wedge\ \forall \vec{x} \forall \vec{X} \dot{\in} Y \ \left(\varphi(\vec{P}, \vec{x}, \vec{X}) \leftrightarrow \varphi^Y(\vec{P}, \vec{x}, \vec{X})\right)\right),$$

i.e. every tuple of sets lies in a coded ω-model such that φ is absolute, at all parameters from the model, between the model and the universe.

Reflection is dependent choice

Σ^1_n formulas: $\exists X_1 \forall X_2 \dots Q X_n \, \theta$ with θ arithmetical.

Σ^1_n-Rfn: the instances of Rfn with $\varphi \in \Sigma^1_n$.

For $\eta(n, X, Y)$ let $(Z)_{<n}$ be the restriction of Z to its columns below n. The *strong dependent choice* scheme strong Σ^1_k-DC (Kreisel; Simpson, §VII.6) consists, for $\eta \in \Sigma^1_k$, of

$$\exists Z \, \forall n \, \big(\exists Y \, \eta\big(n, (Z)_{<n}, Y\big) \to \eta\big(n, (Z)_{<n}, (Z)_n\big)\big).$$

Theorem (essentially Simpson, §VII.7)

Over ACA_0*, for every* $n \geq 1$*:* $\quad \Sigma^1_n$-Rfn $\iff$ *strong* Σ^1_n-DC.

Strength over ACA_0

Known strength of strong dependent choice (Friedman; Simpson, §VII.6):

- $n = 1, 2$: strong Σ^1_n-DC $\iff$ Π^1_n-CA_0;
- all n: strong Σ^1_{n+1}-DC $\implies$ Π^1_{n+1}-CA_0, and is Π^1_4-conservative over it.

Consequently

- Σ^1_{n+1}-Rfn implies Π^1_{n+1}-CA_0 over ACA_0: reflection climbs the whole comprehension hierarchy;
- $\mathsf{ACA}_0 + \mathrm{Rfn}$ contains full Z_2;
- Rfn is unprovable even in Π^1_n-CA_0, for every n.

Main result

Over weak bases the picture reverses: reflection costs nothing.

Theorem

$\mathsf{RCA}_0 + \mathrm{Rfn}$ *is* Π^1_1*-conservative over* RCA_0.

Theorem

$\mathsf{WKL}_0 + \mathrm{Rfn}$ *is* Π^1_1*-conservative over* WKL_0*, and in fact over* RCA_0.

Theorem

$\mathsf{WKL}^*_0 + \mathrm{Rfn}$ *is* Π^1_1*-conservative over* RCA^*_0.

The latter combines the former with results of Fiori-Carones–Kołodziejczyk–Wong–Yokoyama on $\mathsf{WKL}^*_0 + \neg \mathrm{I}\Sigma^0_1$, which entail that this theory proves Rfn.

Consequences: the first-order part of $\mathsf{WKL}_0 + \mathrm{Rfn}$ is exactly $\mathrm{I}\Sigma_1$, and $\mathsf{WKL}_0 + \mathrm{Rfn}$ is Π^0_2-conservative over PRA, i.e. finitistically reducible.

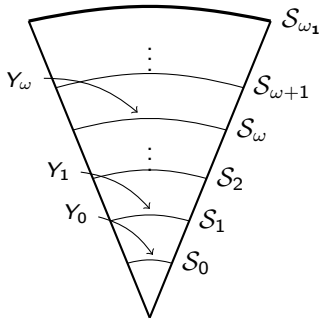
Idea of the proof: an ω_1-tower of forcing extensions

We extend any countable $(M, \mathcal{S}_0) \models \mathrm{RCA}_0$ to $(M, \mathcal{S}_{\omega_1}) \models \mathrm{RCA}_0 + \mathrm{Rfn}$ with the same first-order part $M \Rightarrow$ Π^1_1-conservativity.

$$\mathcal{S}_{\alpha+1} = \{X \mid X \text{ is } \Delta^0_1\text{-def. from } Y_\alpha \text{ in } M\},$$
$$\mathcal{S}_\lambda = \bigcup_{\alpha<\lambda} \mathcal{S}_\alpha \quad \text{for limit } \lambda,$$

where Y_α is a *listing* of $\mathcal{S}_\alpha$ — its sections are exactly the members of $\mathcal{S}_\alpha$ — taken *generic* for the Lévy collapse $\mathcal{P}(\omega) \to \omega$ from the perspective of $(M, \mathcal{S}_\alpha)$.

Validity of Rfn: a modified Löwenheim–Skolem argument whose finite stages are levels $\mathcal{S}_{\alpha_i}$. Given φ with parameters in $\mathcal{S}_{\alpha_0}$, pick $\alpha_0 < \alpha_1 < \cdots$ with $\mathcal{S}_{\alpha_{i+1}}$ containing witnesses for φ's subformulas over $\mathcal{S}_{\alpha_i}$; the result $\mathcal{S}_\lambda$, $\lambda = \lim_{i\to\omega} \alpha_i$, is φ-elementary in $\mathcal{S}_{\omega_1}$, and its listing Y_λ witnesses reflection.



first-order part M fixed

Metamathematics of the proof

The conservativity is a Π^0_2 statement about proofs, but the ω_1-tower proof is not formalizable even in Z_2.

Theorem

$\mathsf{PRA} + 1\text{-}\mathrm{Con}(\mathsf{Z}_2)$ *proves that* $\mathsf{WKL}_0 + \mathsf{Rfn}$ *is* Π^1_1*-conservative over* RCA_0.

Two different properties of a theory $T \supseteq \mathsf{PRA}$:

- *finitistic reducibility*: T is Π^0_2-conservative over PRA;
- *finitistically justifiable* reducibility: the conservation is itself proved by finitary means.

For example, the Π^1_1-conservativity of WKL_0 over RCA_0 is finitistically justifiable (Sieg, Hájek, Avigad, Kohlenbach). $\mathsf{WKL}_0 + \mathsf{Rfn}$ is reducible, but the only known proof costs $1\text{-}\mathrm{Con}(\mathsf{Z}_2)$.

Open problems

- Is the conservativity of the main theorem finitistically provable?
- What is the speed-up of $\mathsf{WKL}_0 + \mathrm{Rfn}$ over WKL_0 for Π^1_1 theorems? Is it primitive recursive?

Precedents for an essential use of $1\text{-}\mathrm{Con}(\mathsf{Z}_2)$:

- Takeuti's fundamental conjecture: cut elimination for the second-order sequent calculus (Tait);
- strong normalization of System F (Girard).

Both are Π^0_2 theorems equivalent over PRA to $1\text{-}\mathrm{Con}(\mathsf{Z}_2)$. Cut elimination yields a conservation theorem parallel to ours: full comprehension without induction for sets (Z_2 minus set induction) is conservative over PA — and there the $1\text{-}\mathrm{Con}(\mathsf{Z}_2)$ price is *necessary*: the speed-up exceeds every Z_2-provably total function.

The use of AI in this research

The paper grew out of an extended interaction with Anthropic's language model Fable 5.

- starting problem: the strength of the collection scheme over RCA_0 and WKL_0;
- genuinely collaborative route: the model proposed constructions, the author supplied insight, direction, and recontextualization;
- the model's key idea: ω_1-chains with universal sets added at each step; the step from Σ^1_1-collection to full Rfn, and the proof of the coding lemma (the construction of the listings Y_α): the author;
- AI-driven Lean formalization: the main theorem for WKL_0 is machine-checked.

Thank you!

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