Degrees of the finite model property

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Problem (Fine, 1974). What is the degree of incompleteness in extensions of K, $\mathsf{K4}$, $\mathsf{S4}$ and IPC?

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A characterization of degrees of incompleteness in extensions of $\mathsf{K4}$, $\mathsf{S4}$ and IPC remains an outstanding open problem.

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As with the degree of incompleteness, all but one of such L' lack the fmp.

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Thus, in the lattice of all normal modal logics the dichotomy holds also for the degrees of fmp.

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Using the Blok-Esakia isomorphism this result generalizes to extensions of $\mathsf{K4}$ and $\mathsf{S4}$.

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Answer: Yes, conisder completeness for finite frames (the fmp) for superintuionistic and transitive modal logics.

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Theorem (J. Aguilera, N.B., and T. Takahashi, 2026)
Each nonzero $\kappa \leq 2^{\aleph_0}$ is realized as the degree of fmp of some superintuitionistic logic (or some transitive modal logic) in ZFC.

Duality for Heyting algebras

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Given a Heyting algebra $\mathfrak{A}$ let $\mathfrak{A}_*$ be the frame of prime filters $(\mathrm{Pf}(\mathfrak{A}), \subseteq)$ with the topology generated by $\{\sigma(a), \sigma(a)^c : a \in \mathfrak{A}\}$, where

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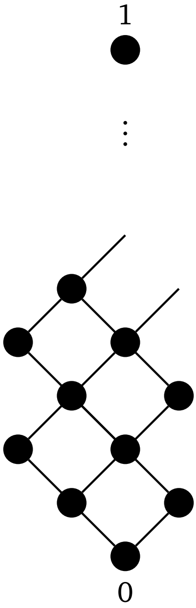
Theorem (Esakia, 1974) For each Heyting algebra $\mathfrak{A}$ we have

$$\mathfrak{A} \cong (\mathfrak{A}_*)^*.$$

The Rieger-Nishimura Lattice



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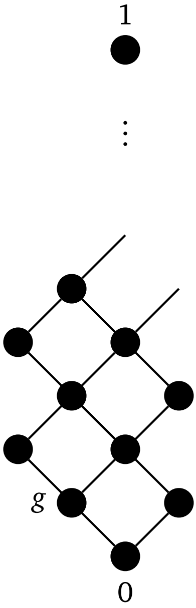

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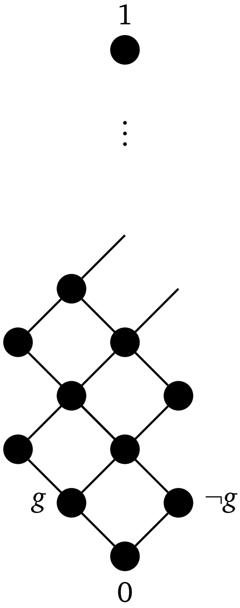

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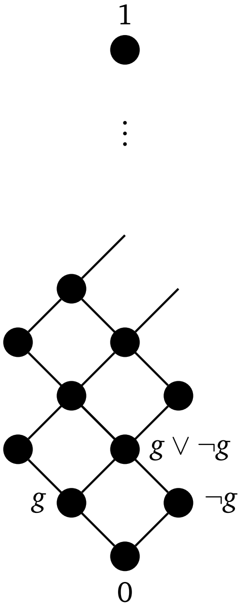

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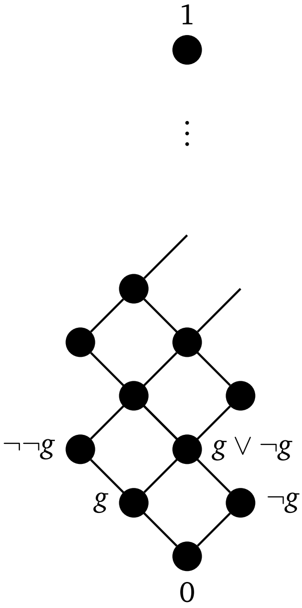

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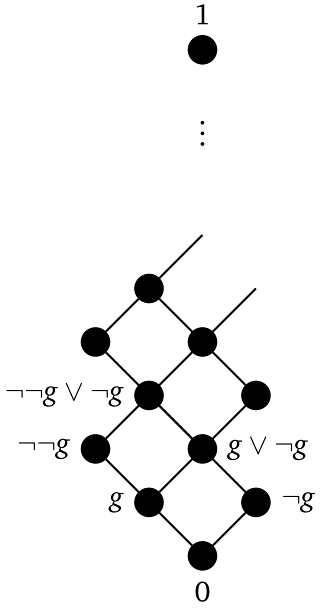

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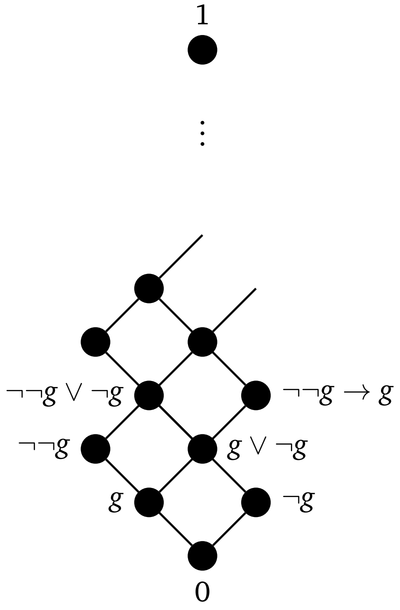
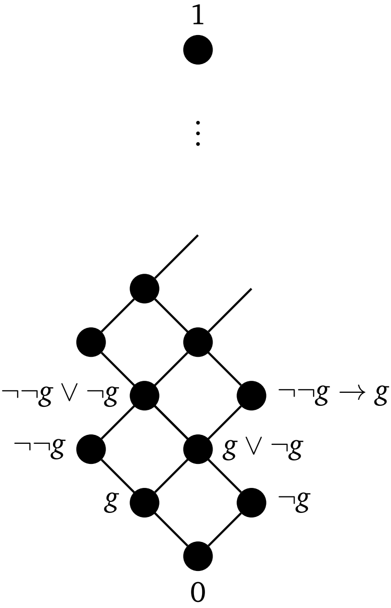


1-generated free Heyting algebra



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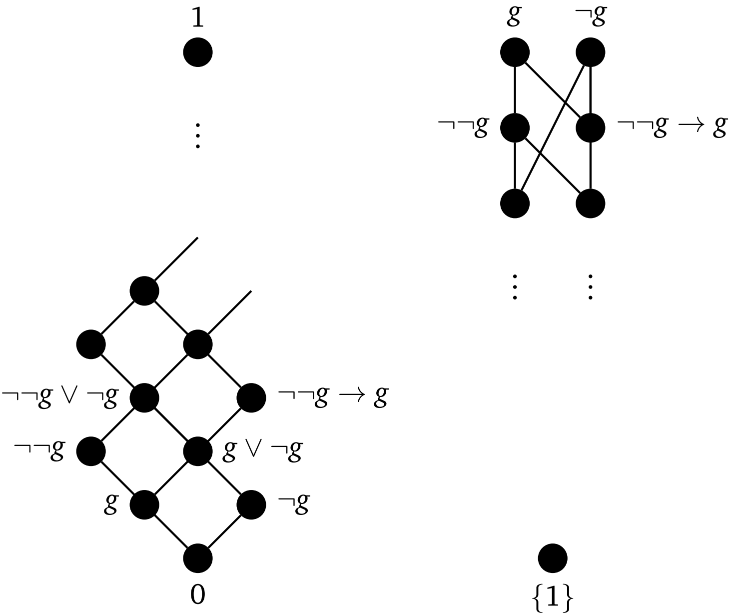

Sums

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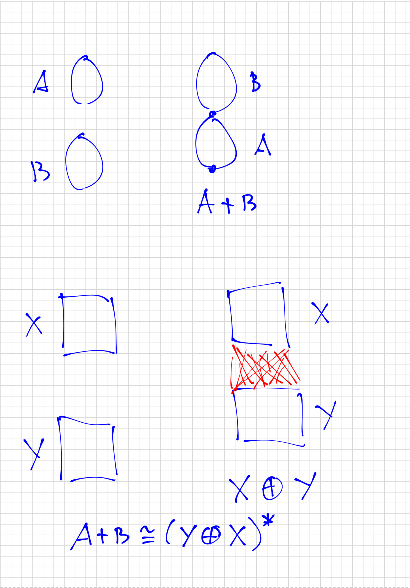
For two frames $\mathfrak{F}$ and $\mathfrak{G}$, we denote by $\mathfrak{F} \oplus \mathfrak{G}$ the frame obtained by pasting $\mathfrak{G}$ below $\mathfrak{F}$.

The Kuznetsov-Gerčiu logic



The Kuznetsov-Gerčiu logic

The Kuznetsov-Gerčiu logic KG is the si-logic of all frames $\mathfrak{F}_1 + \cdots + \mathfrak{F}_n$ where $\mathfrak{F}_1, \ldots, \mathfrak{F}_n$ are generated subframes of RN.

Generated subframes

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2. W' is topologically closed.

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The Rieger-Nishimura ladder $\mathfrak{L}$.

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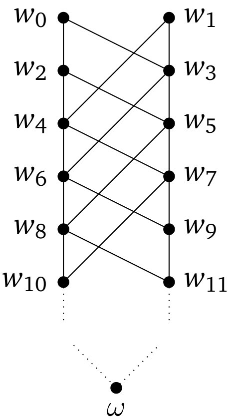
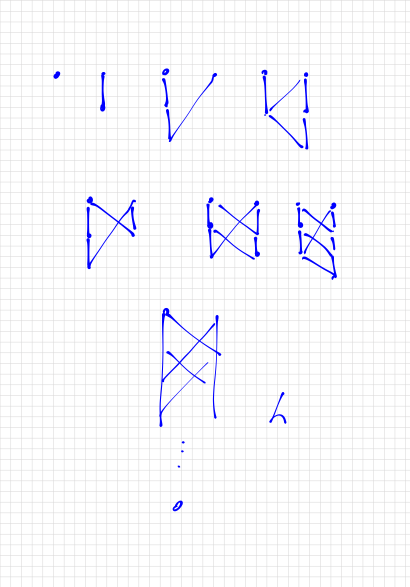
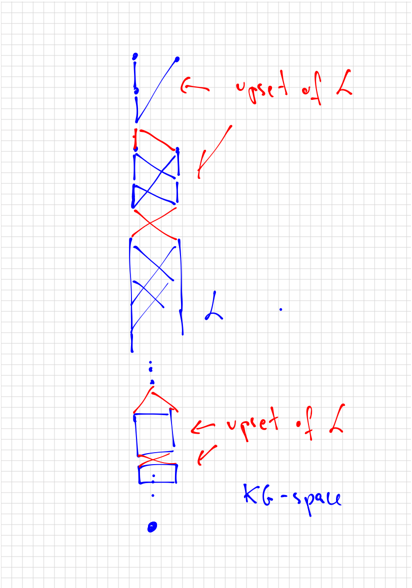


Generated subframes of $\mathcal{L}$



Rooted KG-frames



p-morphic images

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p-morphic image is an onto map under a p-morphism.

Subframe formulas

Subframe Lemma. For each finite rooted $\mathfrak{F}$, there is a formula $\beta(\mathfrak{F})$ such that for each $\mathfrak{G}$ we have

$$\mathfrak{B} \not\models \beta(\mathfrak{F}) \text{ iff } \mathfrak{F} \text{ is a p-morphic image of a subframe of } \mathfrak{G}.$$

The Kuznetsov-Gerčiu logic

Theorem (Kracht, 1993) KG is axiomatized by subframe formulas.

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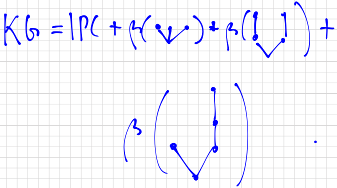

Splittings and Jankov formulas

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An si-logic is join-splitting if it is a join in $\mathrm{Ext}\,\mathsf{IPC}$ of a set of splitting si-logics.

Splittings and Jankov formulas

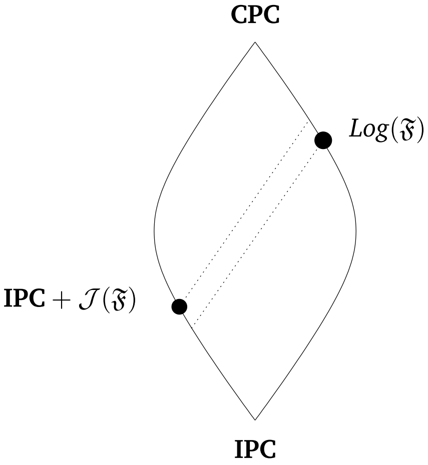

Figure: Splitting of the lattice of varieties of Heyting algebras

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With each finite rooted $\mathfrak{F}$ we can associate the formula (the Jankov formula of $\mathfrak{F}$ and denoted $\mathcal{J}(\mathfrak{F})$) that axiomatizes the least si-logic L such that $\mathfrak{F} \not\models \mathsf{L}$.

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Jankov Lemma. Let $\mathfrak{F}$ and $\mathfrak{G}$ be frames with $\mathfrak{F}$ finite and rooted. Then

$$\mathfrak{G} \not\models \mathcal{J}(\mathfrak{F}) \text{ iff } \mathfrak{F} \text{ is a p-morphic image of a generated subframe of } \mathfrak{G}.$$

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Consequently, L is a join-splitting logic iff L is axiomatizable by Jankov formulas.

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So the degree of fmp above KG agrees with the degree of fmp in si-logics.

The fmp span

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The degree of fmp of L is the cardinality of $\mathrm{fmp}(\mathsf{L})$.

Proof idea of the Antidochotomy theorem for $\kappa < \aleph_0$

Consider the space

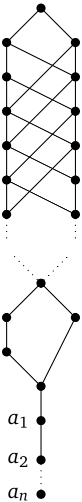

Figure: The poset underlying $\mathfrak{G}_n$.

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We define:

$$\begin{aligned}
\mathsf{L}_0 &= \mathrm{Log}(\mathcal{R}_n) \\
\mathsf{L}_1 &= \mathrm{Log}(\mathcal{R}_n \cup \{\mathfrak{G}_1\}) \\
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Proof idea

Let $\mathcal{R}_n$ be the class of rooted members of $\mathsf{Fin}(\mathrm{Log}(\mathfrak{G}_n))$.

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Main theorem. Let L be an extension of KG. If $\mathsf{L} \in \mathrm{fmp}(\mathrm{Log}(\mathfrak{G}_n))$, then $\mathsf{L} = \mathsf{L}_i$ for some $i = 0, \ldots, n$.

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As KG is axiomatized by Jankov formulas it is enough to consider only logics above KG.

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Let us first characterize generated subframes and p-morphic images of $\mathfrak{L}$.

The Kuznetsov-Gerčiu logic

The Rieger-Nishimura ladder $\mathfrak{L}$.

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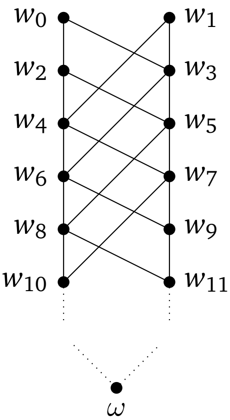
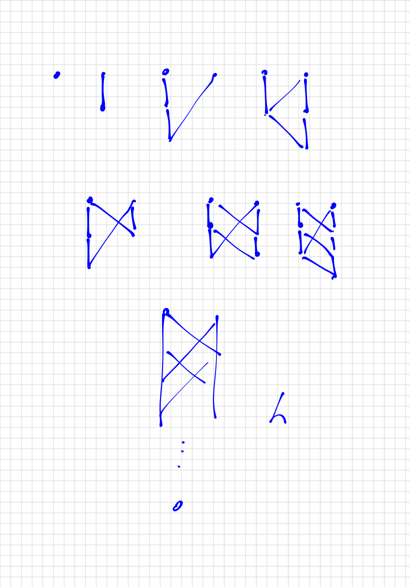
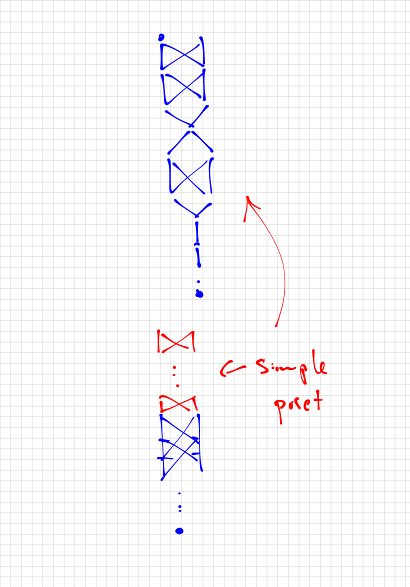


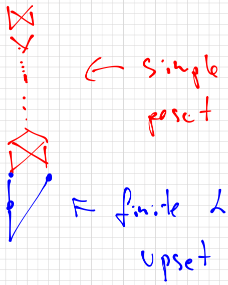
Generated subframes of $\mathcal{L}$



p-morphic images of $\mathcal{L}$



Finite generated subframes of p-morphic images



Proof sketch of the Antidochotomy theorem for $\kappa < \aleph_0$

Consider the space

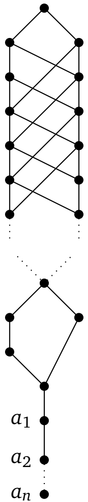


Figure: The poset underlying $\mathfrak{G}_n$.

Finite rooted frames

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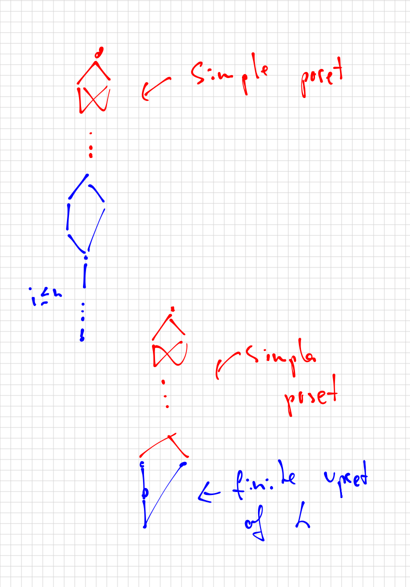
Fin. gen. sub. of p-morphic images of $\mathfrak{G}_n$.



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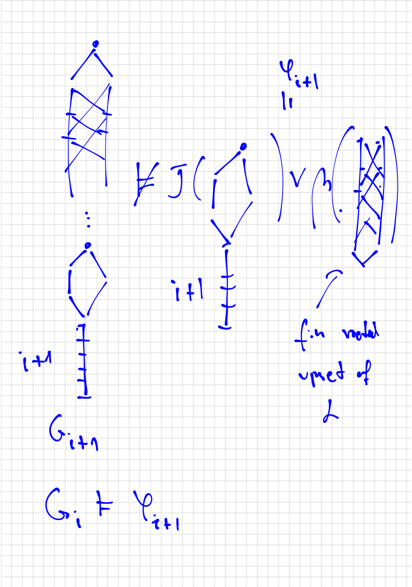
Proof strategy

Lemma

- $\mathsf{L}_0 \supsetneq \cdots \supsetneq \mathsf{L}_n$.
- $\mathsf{Fin}(\mathsf{L}_0) = \cdots = \mathsf{Fin}(\mathsf{L}_n) = \mathcal{R}_n$.

Proof strategy



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Then

$$\mathsf{L} = max\{\mathrm{Log}(\mathcal{R}_n \cup \{X\}) : X \text{ is an } \mathsf{L}\text{-frame}\}) = \mathsf{L}_k$$

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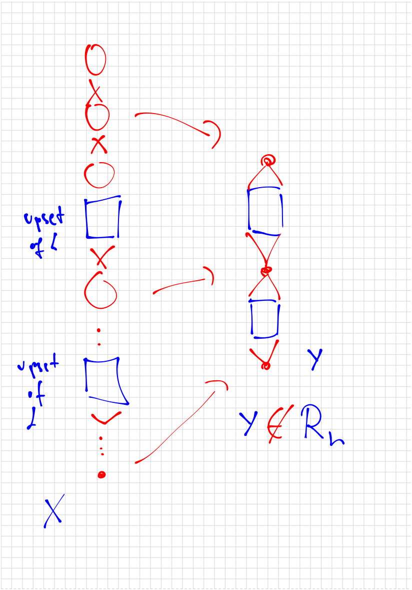
We analyse a few typical cases.

Proof strategy



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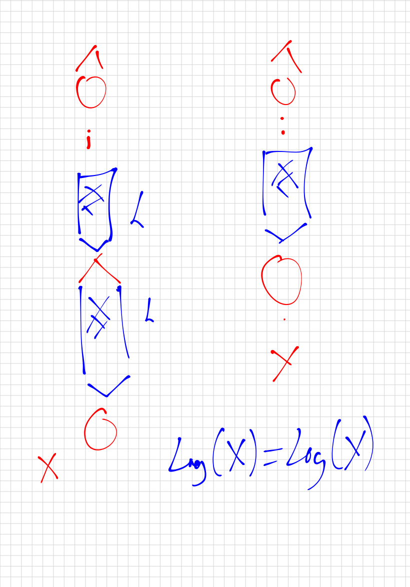

Antidichotomy

Main Result (Antidichotomy Theorem)

For each nonzero cardinal κ such that $\kappa \leq \aleph_0$ or $\kappa = 2^{\aleph_0}$, there is an si-logic L such that $\mathsf{deg}(\mathsf{L}) = \kappa$.

Degree of FMP in transitive modal logics

These results can be extended to transitive modal logic.

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For each nonzero cardinal κ such that $\kappa \leq \aleph_0$ or $\kappa = 2^{\aleph_0}$, there is an extension L of $\mathsf{K4}$ ($\mathsf{S4}$ or Grz) such that $\deg(\mathsf{L}) = \kappa$.

Degrees of fmp

This answers a question posed by Litak (2008):

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Answer: Yes, conisder completeness for finite frames (the fmp) for superintuionistic and transitive modal logics.

Other dichotomy theorems

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Theorem. (Chen, 2025). An extension of tense modal logic K_t, $\mathsf{K4}_t$ or $\mathsf{S4}_t$ has the degree of incompleteness (or fmp) either 1 or $2^{\aleph_0}$.

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Thus, by the results of Descriptive Set Theory, they have the cardinality $\leq \aleph_0$ or $2^{\aleph_0}$.

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This solves Question 6.4 (ii) of Jackson and Lee (TAMS, 2018).

Conclusions and future work

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- One could consider this notion for any other logic or variety of algebras (replacing Kripke frames with finite algebras).
- What is the degree of fmp of fixpoint or interpretability logics.
- What is the degree of incompleteness for these logics, also transitive modal and superintuitionistic logics.

Thank you!