Set-theoretical aspects of proof theory via Turing progressions

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Reflections on Set-Theoretic Reflection, Montseny
A conference in celebration of Joan Bagaria's 60th birthday

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- $\mathrm{PRA} + \mathsf{TI}(\varepsilon_0, \Pi^0_1) \vdash \mathsf{Con}(\mathrm{PA})$

Here $\varepsilon_0 := \mathsf{sup}\{\omega, \omega^\omega, \omega^{\omega^\omega}, \ldots\}$;

$\mathsf{TI}(\varepsilon_0, \Pi^0_1)$ is the axiom scheme

$$\forall \alpha \ (\forall \beta{\prec}\alpha \varphi(\beta) \to \varphi(\alpha)) \to \forall \gamma \varphi(\gamma)$$

with $\prec$ some natural predicate on the natural numbers that defines a well-order of order-type ε_0 on $\mathbb{N}$.

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- Kreisel's pathological ordering

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- Other proof theoretical notions $|U|_{\mathsf{sup}}$, $|U|_{\Pi^0_2}$, $|U|_{\mathsf{TI}}$, . . .

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- Essentially, Schütte, Feferman: $|\mathsf{ATR}_0| = \Gamma_0$

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- Collapsing functions using a "big" ordinal Ω

$$C^{\Omega}(\alpha,\beta) = \begin{cases} \text{Closure of } \beta \cup \{0,\Omega\} \\ \text{under:} \\ +,\ (\gamma \mapsto \omega^{\gamma}) \\ (\gamma \mapsto \psi_{\Omega}(\gamma)) \restriction \alpha \end{cases}$$

$$\psi_{\Omega}(\alpha) = \min\{\rho < \Omega \mid C^{\Omega}(\alpha,\rho) \cap \Omega = \rho\}$$

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- Jäger, Pöhlers: The proof-theoretic ordinal of Kripke-Platek set theory is the Bachmann-Howard ordinal $|\mathrm{KP}| = \psi_\Omega(\varepsilon_{\Omega+1})$

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  - For each Δ_0-function there is an admissible set that is closed under this function, that is,

    For each Δ_0-formula G:

$$(M): \quad \forall x \exists y G(x,y) \to \exists z \big(\mathbf{Ad}(z) \wedge \forall x{\in}z\, \exists y{\in}z\ G(x,y)\big)$$

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$$\xi^\alpha(\delta) = \delta\text{th regular } \pi < M \text{ s.t. } C^M(\alpha,\pi) \cap M = \pi$$

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  "However, I should be a little cautious here as a full proof has not yet been written down, mainly because it taxes the limits of human tolerance."

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- We define $|V|^U_{\Pi^0_1} := \sup\{\alpha \mid U^\alpha \subseteq V\}$

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- (Hamkins, Löwe) True in all forcing extensions: yields S4.2 where the .2 axiom is $\Diamond\Box\varphi \to \Box\Diamond\varphi$ Provided ZFC is consistent

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- It has natural topological semantics though

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- this is extended to all formulas:

$$
\begin{array}{lcl}
[\![\bot]\!]^*_{\mathcal{M}} & = & \varnothing; \\
[\![p]\!]^*_{\mathcal{M}} & = & p^*; \\
[\![\neg\phi]\!]^*_{\mathcal{M}} & = & M \setminus [\![\phi]\!]^*_{\mathcal{M}}; \\
[\![\phi \wedge \psi]\!]^*_{\mathcal{M}} & = & [\![\phi]\!]^*_{\mathcal{M}} \cap [\![\psi]\!]^*_{\mathcal{M}}; \\
[\![\Diamond\phi]\!]^*_{\mathcal{M}} & = & d([\![\phi]\!]^*_{\mathcal{M}}).
\end{array}
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Here $d(Y)$ is the set of accumulation points of Y:
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- For $\mathcal{M} := \langle X, \tau \rangle$ a topological space
- an interpretation $*$ maps any propositional variable p to some subset of X
- this is extended to all formulas:

$$\begin{array}{lcl} [\![\bot]\!]^*_{\mathcal{M}} & = & \varnothing; \\ [\![p]\!]^*_{\mathcal{M}} & = & p^*; \\ [\![\neg\phi]\!]^*_{\mathcal{M}} & = & M \setminus [\![\phi]\!]^*_{\mathcal{M}}; \\ [\![\phi \wedge \psi]\!]^*_{\mathcal{M}} & = & [\![\phi]\!]^*_{\mathcal{M}} \cap [\![\psi]\!]^*_{\mathcal{M}}; \\ [\![\Diamond\phi]\!]^*_{\mathcal{M}} & = & d([\![\phi]\!]^*_{\mathcal{M}}). \end{array}$$

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- $\mathcal{M} \models \varphi$ is defined as $\forall *\, [\![\varphi]\!]^*_{\mathcal{M}} = X$

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- Bagaria, Magidor, Sakai: calibrating the consistency strength of non-discreteness for the topologies τ_ξ corresponding to the $[\xi]$ modality in GLP_Λ

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- Japaridze: The behavior of worms is governed by the simple propositional modal logic GLP

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- Axioms of KP_0: Extensionality, Pair, Union, Infinity, Δ_0-Separation, Δ_0-Collection, Regularity, Transitive Containment (each set is member of a transitive set), and Totality of Rank Function

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- Here, $\mathtt{RFN}$ refers to the following notion of reflection: For each (externally quantified) natural number n, we denote by $\mathsf{RFN}_{\Sigma_n}(\mathsf{X})$ the following principle
  $$\forall \varphi{\in}\Sigma_n \, \forall a \, \exists \alpha{\in}\mathsf{On} \, [V_\alpha \models \varphi(a) \ \Leftrightarrow \ \models_n \varphi(a)].$$
  with $\models_n$ a partial truth predicate

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