Lectures on the Tarski Boundary

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Abstract This short series of lectures is planned as an overview of the main results revolving around the question

Which principles for the notion of truth enable us to derive *new* sentences about natural numbers?

The underlying picture is the following: we start from a theory in the language of arithmetic, called the *base theory*, which encodes some portion of our knowledge about natural numbers. To this theory we add a *truth predicate* for arithmetical sentences, that is we extend the arithmetical signature with a fresh predicate T, and accept certain axioms governing its use. Then we ask whether there are arithmetical sentences which follow from the extended theory but are unprovable in the base theory. If yes, the resulting theory is called *strong* or *non-conservative*. Otherwise, we call it *weak* or *conservative*. The main goal is to trace how the answer changes upon varying the new axioms for T.

More concretely, we shall investigate the properties of *axiomatic truth theories*. These are, by definition, extensions of Peano Arithmetic (PA) which admit a fresh unary predicate T together with some specific axioms characterising T as a truth predicate for the language of arithmetic. The *Tarski Boundary* cuts across the realm of such theories separating conservative extensions of PA from the non-conservative ones. We delineate the contour of the boundary by providing many examples of weak and strong truth theories. In particular we shall comment on when a truth theory is strong enough to reconstruct the usual proof of soundness of PA and prove the *Global Reflection Principle* for PA, that is the sentence

$$\forall \phi \in \mathscr{L}_{\mathrm{PA}}\big(\mathrm{Prov}_{\mathrm{PA}}(\phi) \to T(\phi)\big).$$

The results and techniques we shall talk about have some applications in foundations of mathematics, in particular in the *ordinal analysis*. In the course of the lectures we will explain the basic new ingredients of an approach presented in a recent paper of Lev Beklemishev and Fedor Pakhomov "Reflection algebras and conservation results for theories of iterated truth".

Lecture 1: Introduction to Axiomatic Theories of Truth

Time Monday, 3rd Feb 2020, 10.00 a.m.-12.00 a.m.
Room 4085 (Lluís Vives Room), Philosophy Faculty, C. Montalegre 6

Abstract The idea of the lecture is to give a bird's-eye view on the discipline. In the first part of the lecture we explain some motivations for studying the notion of truth axiomatically. We briefly revisit the classical work of Friedman and Sheard [1987] and Feferman [1991] and spend some time describing the use of a truth predicate in a recent approach to ordinal analysis by Beklemishev and Pakhomov [2019].

In the second part of the lecture we introduce basic types of axiomatizations for the truth predicate and define the canonical

- typed and disquotational theories: TB$(^-)$, UTB$(^-)$.
- typed and compositional theory: CT$(^-)$.
- untyped and compositional theories: FS and KF.
- untyped and disquotational theories: PTB, PUTB.

We summarize their properties, including the results on (ω-)consistency, arithmetical consequences and compatibility with reflection principles. The basic source for this lecture is the textbook by Halbach [2011].

References

Lev Beklemishev and Fedor Pakhomov. Reflection algebras and conservation results for theories of iterated truth. 2019. URL `https://arxiv.org/abs/1908.10302`.

S. Feferman. Reflecting on incompleteness. *Journal of Symbolic Logic*, 56: 1–49, 1991.

Harvey Friedman and Michael Sheard. An axiomatic approach to self-referential truth. *Annals of Pure and Applied Logic*, 33(1):1–21, 1987. doi: 10.1016/0168-0072(87)90073-x.

Volker Halbach. *Axiomatic theories of truth*. Cambridge University Press, 2011.

Lecture 2: Conservativity of Compositional Truth

Time Tuesday, 4th Feb 2020, 10.00 a.m.-12.00 a.m.
Room 4085 (Lluís Vives Room), Philosophy Faculty, C. Montalegre 6

Abstract The aim of the lecture is to carefully present the model-theoretical proof of conservativity of the theory of compositional truth, CT^-, over PA. Our presentation will follow the lines of Visser and Enayat [2015]. By modifying the proof, we draw some conclusions about the regularity properties for the predicate T, which, although unprovable in CT^-, can be conservatively added to this theory. In particular we shall show that CT^- extended with the sentence expressing

"All axioms of induction are true".

is a conservative extension of PA.

Time permitting, we shall study the limitations of the above result by considering various different axiomatizations of PA.

In understanding the Enayat-Visser conservativity proof a basic knowledge about the structure of non-standard models of PA will be helpful. A good source for this is Kaye [1991] (Chapters 1, 2, 6 & 15).

References

Richard Kaye. *Models of Peano Arithmetic*. Clarendon Press, 1991.

Albert Visser and Ali Enayat. New constructions of satisfaction classes. In Kentaro Fujimoto, José Martínez Fernández, Henri Galinon, and Theodora Achourioti, editors, *Unifying the Philosophy of Truth*. Springer Verlag, 2015.

Lecture 3: Many Faces of Global Reflection

Time Thursday, 6th Feb 2020, 10.00 a.m.-12.00 a.m.
Room 4085 (Lluís Vives Room), Philosophy Faculty, C. Montalegre 6

Abstract During the lecture we approximate the Tarski Boundary from above. We study the principles for the truth predicate, which, over CT^-, are equivalent to the Global Reflection Principle for PA. In particular we prove that, over CT^- the following principles are equivalent to the Global Reflection Principle:

- Δ_0-induction for the language with the truth predicate;
- the global reflection principle for pure logic:

$$\forall \phi \in \mathscr{L}_{\mathrm{PA}} \big(\mathrm{Prov}_{\emptyset}(\phi) \to T(\phi)\big);$$

- "The set of true sentences is closed under provability in first-order logic";
- "The set of true sentences is closed under provability in propositional logic";
- the disjunctive correctness principle + "All induction axioms are true".

These results are all summarized in Łełyk [2017]. Time permitting, we shall outline an argument by Fedor Pakhomov (from Enayat and Pakhomov [2019]) that the disjunctive correctness axiom implies the sentence "All axioms of induction are true".

References

Ali Enayat and Fedor Pakhomov. Truth, disjunction, and induction. *Archive for Mathematical Logic*, 58(5-6):753–766, 2019. doi: 10.1007/s00153-018-0657-9.

Mateusz Łełyk. Axiomatic theories of truth, bounded induction and reflection principles, 2017. URL http://depotuw.ceon.pl/bitstream/handle/item/2266/3501-DR-FF-151176.pdf?sequence=1.

Lecture 4: The Arithmetical Part of Global Reflection

Time Friday, 7th Feb 2020, 10.00 a.m.-12.00 a.m.
Room 4085 (Lluís Vives Room), Philosophy Faculty, C. Montalegre 6

Abstract In the lecture we characterize the arithmetical consequences of the extension of CT^- with the Global Reflection Principle for PA and prove that they can be axiomatized by ω-many iterations of uniform reflection over PA. The conservativity part is originally due to Kotlarski Kotlarski [1986] (modulo the results in Smoryński [1977]). We present a model-theoretical proof based on the notion of *prolongable satisfaction classes*. As a byproduct we give model-theoretical characterizations of theories of finite iterations of uniform reflection principle over PA. Finally, we show how, using this machinery, one can obtain a model-theoretical proof of Theorem 1 in Beklemishev and Pakhomov [2019].

In understanding the lecture, some knowledge about the development of model theory in PA might be helpful. Kaye [1991] (Chapter 13.2), Enayat et al. [forthcoming] (Section 2.2) and Kossak and Schmerl [2006] (Chapter 1) contain good introductions. We will introduce all the relevant definitions, but most probably will not have time for a very careful explanation.

References

Lev Beklemishev and Fedor Pakhomov. Reflection algebras and conservation results for theories of iterated truth. 2019. URL https://arxiv.org/abs/1908.10302.

Ali Enayat, Mateusz Łełyk, and Bartosz Wcisło. Truth and feasible reducibility. *Journal of Symbolic Logic*, pages 1–58, forthcoming. doi: 10.1017/jsl.2019.24. URL https://arxiv.org/abs/1902.00392.

Richard Kaye. *Models of Peano Arithmetic*. Clarendon Press, 1991.

Roman Kossak and James Schmerl. *The Structure of Models of Peano Arithmetic*. Clarendon Press, 2006.

H. Kotlarski. *Mathematical Logic Quarterly*, 32:531–534, 1986.

C. Smoryński. $\omega-$consistency and reflection. In *Colloque International de Logique (Colloq. Int. CNRS)*, pages 167–181. CNRS Inst. B. Pascal, 1977.