Boolos' Analytical completeness

Kripke Models

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Provability

- Formulas of arithmetic can be coded within the language of arithmetic. Given a formula ϕ, let $\ulcorner\phi\urcorner$ be its code.
- $\mathrm{Prov}(x)$ is the Σ_1-formula stating that x is the code of a formula that is provable in PA.
- $\mathrm{Prf}(y, x)$ is the Δ_1-formula stating that x is the code of a formula y codes a proof of it in PA.

Notation:

We will write:

- $\mathrm{Prov}(\phi)$ instead of $\mathrm{Prov}(\ulcorner\phi\urcorner)$;
- $\mathrm{Prf}(y, \phi)$ instead of $\mathrm{Prf}(y, \ulcorner\phi\urcorner)$.

Properties of the Prov

Löb's derivability conditions

i. $\mathsf{PA} \vdash \phi \Rightarrow \mathsf{PA} \vdash \mathsf{Prov}(\phi)$;

ii. $\mathsf{PA} \vdash \mathsf{Prov}(\phi \to \psi) \to \big(\mathsf{Prov}(\phi) \to \mathsf{Prov}(\psi)\big)$;

iii. $\mathsf{PA} \vdash \mathsf{Prov}(\phi) \to \mathsf{Prov}\big(\mathsf{Prov}(\phi)\big)$.

Theorem

For every Σ_1-formula ϕ,

$$\mathsf{PA} \vdash \phi \to \mathsf{Prov}(\phi).$$

Löb's Theorem

$\mathsf{PA} \vdash \mathsf{Prov}(\phi) \to \phi \Rightarrow \mathsf{PA} \vdash \phi$.

Gödel - Löb modal logic

GL is a propositional modal logic with the unary modality $\Box$.

Axioms: Boolean tautologies

1. $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$;
2. $\Box\varphi \to \Box\Box\varphi$;
3. $\Box(\Box\varphi \to \varphi) \to \Box\varphi$.

Inference rules:

- *Modus Ponens*;
- *Necessitation*: $\dfrac{\varphi}{\Box\varphi}$.

Solovay's Theorem for GL

An interpretation/realization is a function $(\cdot)^*$ that maps propositional variables to formulas of arithmetic. It can be naturally expanded to a function from all modal formulas:

- $(\phi \to \psi)^* := (\phi)^* \to (\psi)^*$;
- $(\neg\phi)^* := \neg(\phi)^*$;
- $(\Box\phi)^* := \mathsf{Prov}\left((\phi)^*\right)$.

Solovay's Theorem

For every modal formula ϕ:
$\mathsf{GL} \vdash \phi \iff \mathsf{PA} \vdash (\phi)^*$, for every realization $(\cdot)^*$.

Analysis

- In Second order Arithmetic we add set variables and the membership relation $\in$.
- Analysis is the theory of Second order Arithmetic extending PA with the axioms:

IND: $\forall X\ \big(0 \in X \wedge \forall y\ (y \in X \rightarrow y+1 \in X)\big) \rightarrow \forall y\ (y \in X)$;

C: $\exists X \forall y\ \big(y \in X \leftrightarrow \phi(y)\big)$, for every formula ϕ.

- We write $\vdash \phi$ to denote that ϕ is provable in Analysis.

Second order Quantification in the arithmetical hierarchy

- Δ^0_0-formulas are those with only bounded quantification;
- Σ^0_{n+1}-formulas are those equivalent to a formula $\exists x\, \phi(x)$ where $\phi(x)$ is Π^0_n;
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- Π^0_{n+1}-formulas are those equivalent to a formula $\exists x\, \phi(x)$ where $\phi(x)$ is Σ^0_n;
- Π^1_0-formulas are those that are Π^0_n for some n;
- Σ^1_{n+1}-formulas are those equivalent to a formula $Q_1 x_1 \ldots Q_m x_m \exists X\, \phi(x)$ where $\phi(x)$ is Π^1_n;
- Π^1_{n+1}-formulas are those equivalent to a formula $Q_1 x_1 \ldots Q_m x_m \forall X\, \phi(x)$ where $\phi(x)$ is Σ^1_n.

Here the Q_i are one of the $\exists$ and $\forall$.

ω-Provability

- $\omega \vdash \phi$ iff ϕ belongs to the closure of the class of the axioms of analysis under the usual finitary rules and under the ω-rule.
- ω-rule: $\dfrac{\phi(\underline{0}), \phi(\underline{1}), \ldots}{\forall x\, \phi(x)}$
- ω-Prov denotes the class of Godel numbers of ω-provable formulas;
- $\mathrm{Prov}_\omega(x)$ denotes the Π^1_1-formula of analysis defining ω-Prov;

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- ω-Prov denotes the class of Godel numbers of ω-provable formulas;
- $\mathrm{Prov}_\omega(x)$ denotes the Π^1_1-formula of analysis defining ω-Prov;
- $\mathrm{Con}(x) := \neg\,\mathrm{Prov}(\ulcorner\neg\urcorner x)$;
- $\mathrm{Con}_\omega(x) := \neg\,\mathrm{Prov}_\omega(\ulcorner\neg\urcorner x)$.

Japaridze Polymodal Logic

GLP_2 is the propositional modal logic with two unary modalities $[0]$ and $[1]$.

Axioms: Boolean tautologies

Inference rules:

- *Modus Ponens*;
- *Necessitation*: $\dfrac{\varphi}{[i]\varphi}$, for $i = 0, 1$.

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- J1. $[0]\varphi \to [1]\varphi$;
- J2. $\langle 0\rangle\varphi \to [1]\langle 0\rangle\varphi$.

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- *Modus Ponens*;
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Where $\langle i\rangle\phi$ denotes $\neg[i]\neg\phi$.

Solovay in Analysis

We extend the notion of a realization as before to include all modal formulas in the language with the two unary modalities:

- $([0]\phi)^* = \mathrm{Prov}(\phi^*)$;
- $([1]\phi)^* = \mathrm{Prov}_\omega(\phi^*)$.

Theorem

For every modal formula ϕ:

$\mathrm{GLP}_2 \vdash \phi \iff \vdash (\phi)^*$, *for every realization* $(\cdot)^*$.

Arithmetical soundness

Löb's derivability conditions

i. $\vdash \phi \Rightarrow \ \vdash \mathsf{Prov}_\omega(\phi)$;

ii. $\vdash \mathsf{Prov}_\omega(\phi \to \psi) \to \big(\mathsf{Prov}_\omega(\phi) \to \mathsf{Prov}_\omega(\psi)\big)$;

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$\vdash \neg\,\mathsf{Prov}(\phi) \to \mathsf{Prov}_\omega\big(\neg\,\mathsf{Prov}(\phi)\big)$.

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Proof Sketch:

Formalize in Analysis:

$$\frac{\neg\,\mathrm{Prf}(0,\phi), \neg\,\mathrm{Prf}(1,\phi), \ldots}{\forall x\ \neg\,\mathrm{Prf}(x,\phi)}$$

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Π^1_1-completeness

Theorem (Steven Orey; 1956)

For every Π^1_1-formula ϕ,

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Theorem

Every Π^1_1 formula of analysis is equivalent to a formula of the form:

$$\forall f\, \exists x\; R(\overline{f}(x));$$

- *R defines a primitive recursive relation;*
- *$\overline{f}(x)$ denotes the code of the sequence $\langle f(0), \ldots, f(x-1)\rangle$.*

Lemmata we will use to prove Π^1_1-completeness

$Sec = \{s : s \text{ codes a finite sequence and } \omega \vdash \forall f \, \exists x \; R(s * \overline{f}(x))\}$.

Lemma

If $R(s)$, then $s \in Sec$.

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If for every i it holds $s * i \in Sec$, then $s \in Sec$.

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Lemma

If for every i it holds $s * i \in Sec$, then $s \in Sec$.

Proof:

If $\omega \vdash \forall f \, \exists x \; R(s * i * \overline{f}(x))$ for every i,

thus

$$\frac{\forall f \, \exists x \; R(s * 0 * \overline{f}(x)), \forall f \, \exists x \; R(s * 1 * \overline{f}(x)), \ldots}{\forall y \, \forall f \, \exists x \; R(s * y * \overline{f}(x))}$$

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- Let $f(x) = (g(x+1))_x$,

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- Let $f(x) = (g(x+1))_x$,
- then $\forall x\ \neg R(\overline{f}(x))$, a contradiction!

Ignatiev's fragment of GLP_2

$\mathbf{I}_2$ is the subsystem of GLP_2:

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Beklemishev's $\mathbf{J}_2$

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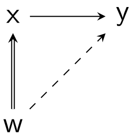
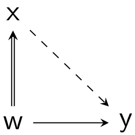
Kripke Models of $\mathbf{I}_2$

An $\mathbf{I}_2$-model M is a quadruple $\langle W, R_0, R_1, V\rangle$ where,

- W is a finite set;
- $V \subseteq W \times \{\text{sentence letters}\}$;
- $R_0, R_1 \subseteq W \times W$ are transitive, irreflexive, and for all $w, x, y \in W$:
  - if wR_1x and xR_0y then wR_0y;
  - if wR_1x and wR_0y then xR_0y.





Model completeness for $\mathbf{I}_2$

The relation $\vDash$ is defined as per usual with:

- $M, x \vDash p$ iff xVp;
- $M, x \vDash [0]\phi$ iff $M, y \vDash \phi$ for every y such that xR_0y;
- $M, x \vDash [1]\phi$ iff $M, y \vDash \phi$ for every y such that xR_1y;

Theorem

$\mathbf{I}_2 \vdash \phi$ iff for every $\mathbf{I}_2$-model $M = \langle W, R_0, R_1, V\rangle$, and every $x \in W$,
it holds $M, x \vDash \phi$.

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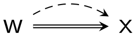
Where GLP_2 fails:



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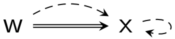
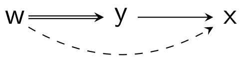


Two relations

We define the following two relations:

- $wR_{\geq 0}x$ iff $wR_0x \vee wR_1x \vee \exists\, y\ (wR_0yR_1x)$;



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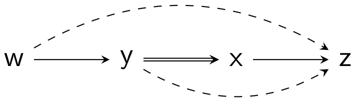

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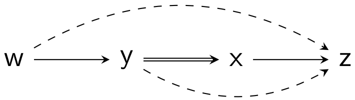
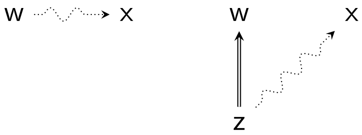


- $w\hat{R}_0x$ iff $wR_{\geq 0}x \vee \exists\, z\ (zR_1w \wedge zR_{\geq 0}x)$;



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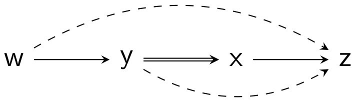

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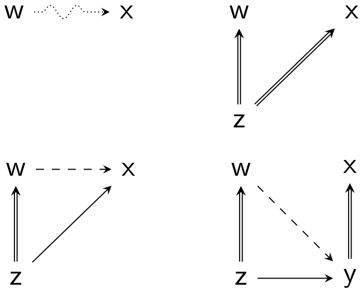

Definition:

Given an $\mathbf{I}_2$-model $M = \langle W, R_0, R_1, V \rangle$ and $w \in W$, the generated submodel of M at w is the model:

$$w{\uparrow}M = \langle W_w, R_0 \cap W_w^2, R_1 \cap W_w^2, V \restriction W_w \rangle,$$

where $W_w = \{x \in W : wR_{\geq 0}x\} \cup \{w\}$.

ϕ-completeness of am $\mathbf{I}_2$-model

Definitions

- $M = \langle W, R_0, R_1, V \rangle$ is ϕ-complete iff for every $x \in W$, $M, x \vDash [0]\psi \to [1]\psi$ for all subsentences $[0]\psi$ of ϕ.
- $\Delta\phi$ is $\phi \wedge [0]\phi \wedge [1]\phi \wedge [0][1]\phi$;
- $M\phi$ is $\bigwedge\{\Delta([0]\psi \to [1]\psi) : [0]\psi$ is a subsentence of $\phi\}$.

Lemma

$w{\uparrow}M$ is ϕ-complete iff $M, w \vDash M\phi$.

Theorem

Assume $M = \langle W, R_0, R_1, V \rangle$ *is* ϕ*-complete and let* $N = \langle W, \hat{R}_0, R_1, V \rangle$. *For every subsentence* ψ *of* ϕ *and* $w \in W$,

$$M, w \vDash \psi \text{ iff } N, w \vDash \psi.$$

Theorem

Assume $M = \langle W, R_0, R_1, V \rangle$ is ϕ-complete and let $N = \langle W, \hat{R}_0, R_1, V \rangle$. For every subsentence ψ of ϕ and $w \in W$,

$$M, w \vDash \psi \text{ iff } N, w \vDash \psi.$$

Proof

Suppose that $M, w \vDash [0]\psi$ and $N, w \nvDash [0]\psi$.So $N, x \nvDash \psi$ for some $w\hat{R}_0x$ and by the I.H., $M, x \nvDash \psi$. Thus not wR_0x.

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$$M, w \vDash \psi \text{ iff } N, w \vDash \psi.$$

Proof

Suppose that $M, w \vDash [0]\psi$ and $N, w \nvDash [0]\psi$.So $N, x \nvDash \psi$ for some $w\hat{R}_0x$ and by the I.H., $M, x \nvDash \psi$. Thus not wR_0x.

- If wR_1x
- if wR_0yR_1x
- If zR_1w and zR_1x

Theorem

Assume $M = \langle W, R_0, R_1, V\rangle$ is ϕ-complete and let $N = \langle W, \hat{R}_0, R_1, V\rangle$. For every subsentence ψ of ϕ and $w \in W$,

$$M, w \vDash \psi \text{ iff } N, w \vDash \psi.$$

Proof

Suppose that $M, w \vDash [0]\psi$ and $N, w \nvDash [0]\psi$.So $N, x \nvDash \psi$ for some $w\hat{R}_0x$ and by the I.H., $M, x \nvDash \psi$. Thus not wR_0x.

- If wR_1x then $M, w \nvDash [1]\psi$;
- if wR_0yR_1x then $M, y \nvDash [1]\psi$;
- If zR_1w and zR_1x then $M, z \nvDash [1]\psi$;

Theorem

Assume $M = \langle W, R_0, R_1, V \rangle$ is ϕ-complete and let $N = \langle W, \hat{R}_0, R_1, V \rangle$. For every subsentence ψ of ϕ and $w \in W$,

$$M, w \models \psi \text{ iff } N, w \models \psi.$$

Proof

Suppose that $M, w \models [0]\psi$ and $N, w \not\models [0]\psi$.So $N, x \not\models \psi$ for some $w\hat{R}_0x$ and by the I.H., $M, x \not\models \psi$. Thus not wR_0x.

- If wR_1x then $M, w \not\models [1]\psi$; then $M, w \not\models [0]\psi$;
- if wR_0yR_1x then $M, y \not\models [1]\psi$; then $M, y \not\models [0]\psi$;
- If zR_1w and zR_1x then $M, z \not\models [1]\psi$; then $M, z \not\models [0]\psi$;

Theorem

Assume $M = \langle W, R_0, R_1, V\rangle$ *is* ϕ*-complete and let* $N = \langle W, \hat{R}_0, R_1, V\rangle$*. For every subsentence* ψ *of* ϕ *and* $w \in W$*,*

$$M, w \vDash \psi \text{ iff } N, w \vDash \psi.$$

Proof

Suppose that $M, w \vDash [0]\psi$ and $N, w \nvDash [0]\psi$.So $N, x \nvDash \psi$ for some $w\hat{R}_0x$ and by the I.H., $M, x \nvDash \psi$. Thus not wR_0x.

- If wR_1x then $M, w \nvDash [1]\psi$; then $M, w \nvDash [0]\psi$;
- if wR_0yR_1x then $M, y \nvDash [1]\psi$; then $M, y \nvDash [0]\psi$;
- If zR_1w and zR_1x then $M, z \nvDash [1]\psi$; then $M, z \nvDash [0]\psi$;

Thus $M, a \nvDash \psi$ for some wR_0a, a contradiction!

Thank You