Boolos' Analytical completeness

The Solovay Proof

Konstantinos Papafilippou

Ghent University

February 4, 2021

Japaridze Polymodal Logic

GLP_2 is the propositional modal logic with two unary modalities [0] and [1].

Axioms: Boolean tautologies

- L1. $[i](\varphi \to \psi) \to ([i]\varphi \to [i]\psi)$, for $i = 0, 1$;
- L2. $[i]\varphi \to [i][i]\varphi$, for $i = 0, 1$;
- L3. $[i]([i]\varphi \to \varphi) \to [i]\varphi$, for $i = 0, 1$;
- J1. $[0]\varphi \to [1]\varphi$;
- J2. $\langle 0\rangle\varphi \to [1]\langle 0\rangle\varphi$.

Inference rules:

- *Modus Ponens*;
- *Necessitation*: $\dfrac{\varphi}{[i]\varphi}$, for $i = 0, 1$.

Where $\langle i\rangle\phi$ denotes $\neg[i]\neg\phi$.

Ignatiev's fragment of GLP_2

$\mathbf{I}_2$ is the subsystem of GLP_2:

Axioms: Boolean tautologies

L1. $[i](\varphi \to \psi) \to ([i]\varphi \to [i]\psi)$, for $i = 0, 1$;

L2. $[i]\varphi \to [i][i]\varphi$, for $i = 0, 1$;

L3. $[i]([i]\varphi \to \varphi) \to [i]\varphi$, for $i = 0, 1$;

I1. $[0]\varphi \to [1][0]\varphi$;

J2. $\langle 0\rangle\varphi \to [1]\langle 0\rangle\varphi$.

Inference rules:

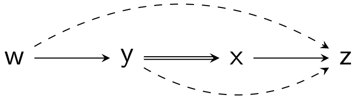
- *Modus Ponens*;
- *Necessitation*: $\dfrac{\varphi}{[i]\varphi}$, for $i = 0, 1$.

Where $\langle i\rangle\phi$ denotes $\neg[i]\neg\phi$.

Two relations

We define the following two relations:

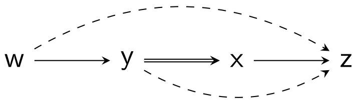
- $wR_{\geq 0}x$ iff $wR_0x \vee wR_1x \vee \exists\, y\ (wR_0yR_1x)$;



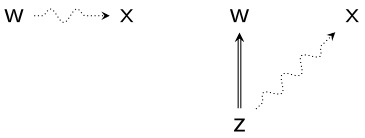
Two relations

We define the following two relations:

- $wR_{\geq 0}x$ iff $wR_0x \vee wR_1x \vee \exists\, y\ (wR_0yR_1x)$;



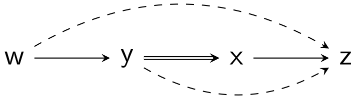
- $w\hat{R}_0x$ iff $wR_{\geq 0}x \vee \exists\, z\ (zR_1w \wedge zR_{\geq 0}x)$;



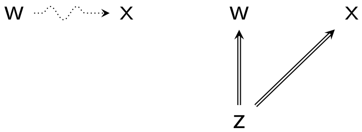
Two relations

We define the following two relations:

- $wR_{\geq 0}x$ iff $wR_0x \vee wR_1x \vee \exists\, y\ (wR_0yR_1x)$;



- $w\hat{R}_0x$ iff $wR_{\geq 0}x \vee \exists\, z\ (zR_1w \wedge zR_{\geq 0}x)$; iff $wR_{\geq 0}x \vee \exists\, z\ (zR_1w \wedge zR_1x)$.



Definitions

- $M = \langle W, R_0, R_1, V \rangle$ is ϕ-complete iff for every $x \in W$, $M, x \vDash [0]\psi \rightarrow [1]\psi$ for all subsentences $[0]\psi$ of ϕ.
- $\Delta\phi$ is $\phi \wedge [0]\phi \wedge [1]\phi \wedge [0][1]\phi$;
- $M\phi$ is $\bigwedge\{\Delta([0]\psi \rightarrow [1]\psi) : [0]\psi$ is a subsentence of $\phi\}$.

Lemma

$w{\uparrow}M$ is ϕ-complete iff $M, w \vDash M\phi$.

Theorem

Assume $M = \langle W, R_0, R_1, V \rangle$ is ϕ-complete and let $N = \langle W, \hat{R}_0, R_1, V \rangle$. For every subsentence ψ of ϕ and $w \in W$,

$$M, w \vDash \psi \text{ iff } N, w \vDash \psi.$$

Analytical Completeness

Suppose $\mathrm{GLP}_2 \nvdash \phi$. Then $\mathbf{I}_2 \nvdash M\phi \to \phi$.

- By the completeness of $\mathbf{I}_2$, there is a model $M = \langle W, R_0, R_1, V\rangle$ and a world e such that $M, e \vDash M\phi$ and $M, e \nvDash \phi$.
- By the generated submodel theorem we may assume that $M = e{\uparrow}M$ and so that M is ϕ-complete.
- Assume that $W = \{1, \ldots, n\}$ and $e = 1$.
- Finally, we add a world 0 and extend R_0 so that $0R_0x$ for all $x \in W$.

Solovay conditions

1. $\vdash \bigvee_{x \in W \cup \{0\}} S_x$;
2. $\vdash \neg(S_x \wedge S_y)$, for every $x \neq y$;

3.i. $\vdash S_w \rightarrow \text{Con}(S_x)$, for every wR_0x;

3.ii. $\vdash S_w \rightarrow \text{Con}_\omega(S_x)$, for every wR_1x;

4.i. $\vdash S_w \rightarrow \text{Prov}\left(\bigvee_{w\hat{R}_0x} S_x\right)$, for every $w \neq 0$;

4.ii. $\vdash S_w \rightarrow \text{Prov}_\omega\left(\bigvee_{wR_1x} S_x\right)$, for every $w \neq 0$.

Solovay conditions

1. $\vdash \bigvee_{x \in W \cup \{0\}} S_x$;
2. $\vdash \neg(S_x \wedge S_y)$, for every $x \neq y$;

3.i. $\vdash S_w \to \mathrm{Con}(S_x)$, for every wR_0x;

3.ii. $\vdash S_w \to \mathrm{Con}_\omega(S_x)$, for every wR_1x;

4.i. $\vdash S_w \to \mathrm{Prov}\left(\bigvee_{w\hat{R}_0x} S_x\right)$, for every $w \neq 0$;

4.ii. $\vdash S_w \to \mathrm{Prov}_\omega\left(\bigvee_{wR_1x} S_x\right)$, for every $w \neq 0$.

Lemma

If $w \in W$, then $\vdash S_w \to \mathrm{Prov}_\omega(\neg S_w)$.

Solovay conditions

1. $\vdash \bigvee_{x \in W \cup \{0\}} S_x$;
2. $\vdash \neg(S_x \wedge S_y)$, for every $x \neq y$;

3.i. $\vdash S_w \rightarrow \mathrm{Con}(S_x)$, for every wR_0x;

3.ii. $\vdash S_w \rightarrow \mathrm{Con}_\omega(S_x)$, for every wR_1x;

4.i. $\vdash S_w \rightarrow \mathrm{Prov}\left(\bigvee_{w\hat{R}_0x} S_x\right)$, for every $w \neq 0$;

4.ii. $\vdash S_w \rightarrow \mathrm{Prov}_\omega\left(\bigvee_{wR_1x} S_x\right)$, for every $w \neq 0$.

Lemma

If $w \in W$, then $\vdash S_w \rightarrow \mathrm{Prov}_\omega(\neg S_w)$.

Proof

By 2., $S_x \rightarrow \neg S_w$ for every wR_1x. Thus $\bigvee_{wR_1x} S_x \rightarrow \neg S_w$.
The proof concludes with (4.ii) and the properties of Prov_ω.

Solovay Proof

Define $(\cdot)^*$ such that $p^* = \bigvee_{wVp} S_w$.

Theorem

For every subsentence ψ of ϕ and $w \in W$:

(a) *if $M, w \vDash \psi$ then $\vdash S_w \to \psi^*$;*

(b) *if $M, w \nvDash \psi$ then $\vdash S_w \to \neg\psi^*$.*

Solovay Proof

Define $(\cdot)^*$ such that $p^* = \bigvee_{wVp} S_w$.

Theorem

For every subsentence ψ of ϕ and $w \in W$:

(a) *if $M, w \vDash \psi$ then $\vdash S_w \to \psi^*$;*

(b) *if $M, w \nvDash \psi$ then $\vdash S_w \to \neg\psi^*$.*

Proof

Assume $\psi = p$, then (a) is trivial and (b) is given by condition 2.:

$$\vdash \neg(S_x \wedge S_y), \text{ for every } x \neq y.$$

Solovay Proof

Proof

Suppose $\psi = [0]\xi$.

Solovay Proof

Proof

Suppose $\psi = [0]\xi$.

(a) Let $M, w \vDash [0]\xi$,

Solovay Proof

Proof

Suppose $\psi = [0]\xi$.

(a) Let $M, w \vDash [0]\xi$, then $M, x \vDash \xi$ for all $w\hat{R}_0 x$.

Solovay Proof

Proof

Suppose $\psi = [0]\xi$.

(a) Let $M, w \vDash [0]\xi$, then $M, x \vDash \xi$ for all $w\hat{R}_0x$.

By the I.H., $\vdash S_x \rightarrow \xi^*$ for all $w\hat{R}_0x$.

Solovay Proof

Proof

Suppose $\psi = [0]\xi$.

(a) Let $M, w \vDash [0]\xi$, then $M, x \vDash \xi$ for all $w\hat{R}_0 x$.

By the I.H., $\vdash S_x \to \xi^*$ for all $w\hat{R}_0 x$.

So $\vdash \bigvee_{w\hat{R}_0 x} S_x \to \xi^*$ and thus $\vdash \text{Prov}\left(\bigvee_{w\hat{R}_0 x} S_x\right) \to \psi^*$.

Solovay Proof

Proof

Suppose $\psi = [0]\xi$.

(a) Let $M, w \vDash [0]\xi$, then $M, x \vDash \xi$ for all $w\hat{R}_0 x$.

By the I.H., $\vdash S_x \rightarrow \xi^*$ for all $w\hat{R}_0 x$.

So $\vdash \bigvee_{w\hat{R}_0 x} S_x \rightarrow \xi^*$ and thus $\vdash \text{Prov}\left(\bigvee_{w\hat{R}_0 x} S_x\right) \rightarrow \psi^*$.

By 4.i., $\vdash S_w \rightarrow \text{Prov}\left(\bigvee_{w\hat{R}_0 x} S_x\right)$ and so $\vdash S_w \rightarrow \psi^*$.

Solovay Proof

Proof

Suppose $\psi = [0]\xi$.

(a) Let $M, w \vDash [0]\xi$, then $M, x \vDash \xi$ for all $w\hat{R}_0 x$.
By the I.H., $\vdash S_x \to \xi^*$ for all $w\hat{R}_0 x$.
So $\vdash \bigvee_{w\hat{R}_0 x} S_x \to \xi^*$ and thus $\vdash \text{Prov}\left(\bigvee_{w\hat{R}_0 x} S_x\right) \to \psi^*$.
By 4.i., $\vdash S_w \to \text{Prov}\left(\bigvee_{w\hat{R}_0 x} S_x\right)$ and so $\vdash S_w \to \psi^*$.

(b) If $M, w \nvDash [0]\xi$

Solovay Proof

Proof

Suppose $\psi = [0]\xi$.

(a) Let $M, w \vDash [0]\xi$, then $M, x \vDash \xi$ for all $w\hat{R}_0x$.

By the I.H., $\vdash S_x \rightarrow \xi^*$ for all $w\hat{R}_0x$.

So $\vdash \bigvee_{w\hat{R}_0x} S_x \rightarrow \xi^*$ and thus $\vdash \mathrm{Prov}\left(\bigvee_{w\hat{R}_0x} S_x\right) \rightarrow \psi^*$.

By 4.i., $\vdash S_w \rightarrow \mathrm{Prov}\left(\bigvee_{w\hat{R}_0x} S_x\right)$ and so $\vdash S_w \rightarrow \psi^*$.

(b) If $M, w \nvDash [0]\xi$ then $M, x \nvDash \xi$ for some wR_0x.

Solovay Proof

Proof

Suppose $\psi = [0]\xi$.

(a) Let $M, w \vDash [0]\xi$, then $M, x \vDash \xi$ for all $w\hat{R}_0x$.
By the I.H., $\vdash S_x \to \xi^*$ for all $w\hat{R}_0x$.
So $\vdash \bigvee_{w\hat{R}_0x} S_x \to \xi^*$ and thus $\vdash \mathrm{Prov}\left(\bigvee_{w\hat{R}_0x} S_x\right) \to \psi^*$.
By 4.i., $\vdash S_w \to \mathrm{Prov}\left(\bigvee_{w\hat{R}_0x} S_x\right)$ and so $\vdash S_w \to \psi^*$.

(b) If $M, w \nvDash [0]\xi$ then $M, x \nvDash \xi$ for some wR_0x.
By the I.H., $\vdash S_x \to \neg\xi^*$.

Solovay Proof

Proof

Suppose $\psi = [0]\xi$.

(a) Let $M, w \models [0]\xi$, then $M, x \models \xi$ for all $w\hat{R}_0x$.
By the I.H., $\vdash S_x \to \xi^*$ for all $w\hat{R}_0x$.
So $\vdash \bigvee_{w\hat{R}_0x} S_x \to \xi^*$ and thus $\vdash \mathrm{Prov}\left(\bigvee_{w\hat{R}_0x} S_x\right) \to \psi^*$.
By 4.i., $\vdash S_w \to \mathrm{Prov}\left(\bigvee_{w\hat{R}_0x} S_x\right)$ and so $\vdash S_w \to \psi^*$.

(b) If $M, w \not\models [0]\xi$ then $M, x \not\models \xi$ for some wR_0x.
By the I.H., $\vdash S_x \to \neg\xi^*$.
Thus $\vdash \mathrm{Con}(S_x) \to \neg\psi^*$.

Solovay Proof

Proof

Suppose $\psi = [0]\xi$.

(a) Let $M, w \vDash [0]\xi$, then $M, x \vDash \xi$ for all $w\hat{R}_0 x$.
By the I.H., $\vdash S_x \to \xi^*$ for all $w\hat{R}_0 x$.
So $\vdash \bigvee_{w\hat{R}_0 x} S_x \to \xi^*$ and thus $\vdash \text{Prov}\left(\bigvee_{w\hat{R}_0 x} S_x\right) \to \psi^*$.
By 4.i., $\vdash S_w \to \text{Prov}\left(\bigvee_{w\hat{R}_0 x} S_x\right)$ and so $\vdash S_w \to \psi^*$.

(b) If $M, w \nvDash [0]\xi$ then $M, x \nvDash \xi$ for some $wR_0 x$.
By the I.H., $\vdash S_x \to \neg\xi^*$.
Thus $\vdash \text{Con}(S_x) \to \neg\psi^*$.
By 3.i., $\vdash S_w \to \text{Con}(S_x)$ and $\vdash S_w \to \neg\psi^*$.

Proving completeness

Therefore, $\vdash S_1 \to \neg\phi^*$;

Proving completeness

Therefore, $\vdash S_1 \rightarrow \neg\phi^*$;

$\vdash \text{Con}(S_1) \rightarrow \neg\,\text{Prov}(\phi^*)$;

3.i. gives $\vdash S_0 \rightarrow \text{Con}(S_1)$;

thus $\vdash S_0 \rightarrow \neg\,\text{Prov}(\phi^*)$;

Proving completeness

Therefore, $\vdash S_1 \rightarrow \neg\phi^*$;

$\vdash \mathrm{Con}(S_1) \rightarrow \neg\,\mathrm{Prov}(\phi^*)$;

3.i. gives $\vdash S_0 \rightarrow \mathrm{Con}(S_1)$;

thus $\vdash S_0 \rightarrow \neg\,\mathrm{Prov}(\phi^*)$;

since $\mathbb{N} \models S_0$, then $\mathbb{N} \models \neg\,\mathrm{Prov}(\phi^*)$ and so $\nvdash \phi^*$.

Requirements for the conditions

Theorem

There is a Π^1_1*-relation* $\rho(x, y)$ *with domain* ω-Prov *such that* $\{\langle x, y\rangle : x, y \in \omega\text{-Prov} \wedge \rho(x, y)\}$ *reflexively well-orders* ω-Prov*; moreover, if* $x \in \omega$-Prov *and* $y \notin \omega$-Prov*, then* $\rho(x, y)$*.*

Notation

Let $\hat{\rho}(\phi, \psi)$ denote the formula $\rho(\neg\phi, \neg\psi)$.

Remark

$\vdash \rho(x, y) \rightarrow \text{Prov}_\omega(x)$.

Therefore, $\vdash \hat{\rho}(x, y) \rightarrow \text{Prov}_\omega(\neg x)$.

OK Functions

Definition

A function $h : \{0, \dots m\} \to W \cup \{0\}$ is w-*OK* iff:

- $h(0) = 0$;
- $h(m) = w$;
- for all $i < m$, either $h(i)\, R_0\, h(i{+}1)$ or $h(i)\, R_1\, h(i{+}1)$;
- there is no i such that $h(i)\, R_1\, h(i{+}1)\, R_0\, h(i{+}2)$.

The function h is *OK* if it is w-*OK* for some $w \in W \cup \{0\}$.

Notation

There is a least $k \leq m$ such that $h(i)\, R_1\, h(i{+}1)$ for all $i \geq k$.
We will denote that k as l_0 and m as l_1.

$$h(0) \rightsquigarrow h(l_0) \rightsquigarrow h(l_1)$$

Solovay sentences

By the generalized diagonal lemma, there are sentences $S_0, S_1 \ldots, S_n$ such that for each $w \in W \cup \{0\}$:

$$\vdash S_w \leftrightarrow w = w \wedge \bigvee \{A_h \wedge B_h \wedge C_h \wedge D_h : h \text{ is } w\text{-}OK\}.$$

- A_h is $\bigwedge_{i<l_0} \bigwedge_{h(i)R_0x} \exists b(\,\mathrm{Prf}(b, \neg S_{h(i+1)}) \wedge \forall a < b \neg\,\mathrm{Prf}(a, \neg S_x));$
- B_h is $\bigwedge_{h(l_0)R_0x} \mathrm{Con}(S_x);$
- C_h is $\bigwedge_{l_0 \leq i < l_1} \bigwedge_{h(i)R_1x} \hat{\rho}(S_{h(i+1)}, S_x);$
- D_h is $\bigwedge_{h(l_1)R_1x} \mathrm{Con}_\omega(S_x).$

Solovay sentences

By the generalized diagonal lemma, there are sentences $S_0, S_1 \ldots, S_n$ such that for each $w \in W \cup \{0\}$:

$$\vdash S_w \leftrightarrow w = w \wedge \bigvee\{A_h \wedge B_h \wedge C_h \wedge D_h : h \text{ is } w\text{-}OK\}.$$

- A_h is $\bigwedge_{i<l_0} \bigwedge_{h(i)R_0x} \exists b(\,\mathrm{Prf}(b, \neg S_{h(i+1)}) \wedge \forall a < b \neg\,\mathrm{Prf}(a, \neg S_x));$
- B_h is $\bigwedge_{h(l_0)R_0x} \mathrm{Con}(S_x);$
- C_h is $\bigwedge_{l_0 \le i < l_1} \bigwedge_{h(i)R_1x} \hat{\rho}(S_{h(i+1)}, S_x);$
- D_h is $\bigwedge_{h(l_1)R_1x} \mathrm{Con}_\omega(S_x).$

Notation

We will write AB_h instead of $A_h \wedge B_h$ etc.

Proving the conditions

Lemma

Suppose wR_0x, then $\vdash S_w \to \mathrm{Con}(S_x)$.

Proving the conditions

Lemma

Suppose wR_0x, then $\vdash S_w \to \mathrm{Con}(S_x)$.

Proof

Let h be w-OK. Then $h(l_0)R_0x$ and so $B_h \to \mathrm{Con}(S_x)$.

Proving the conditions

Lemma

Suppose wR_0x, then $\vdash S_w \to \mathrm{Con}(S_x)$.

Proof

Let h be w-OK. Then $h(l_0)R_0x$ and so $B_h \to \mathrm{Con}(S_x)$.

Lemma

Suppose wR_1x, then $\vdash S_w \to \mathrm{Con}_\omega(S_x)$.

Proving the conditions

Lemma

Suppose wR_0x, then $\vdash S_w \rightarrow \mathrm{Con}(S_x)$.

Proof

Let h be w-OK. Then $h(l_0)R_0x$ and so $B_h \rightarrow \mathrm{Con}(S_x)$.

Lemma

Suppose wR_1x, then $\vdash S_w \rightarrow \mathrm{Con}_\omega(S_x)$.

Notation

- h is R_1-free if $l_0 = l_1$.
- h' R_0-extends h if $l_1 < l'_1$, for every $l_1 \leq i < l'_1$ it holds that $h(i)\, R_0\, h(i{+}1)$ and $h' \upharpoonright dom(h) = h$.

Proving the conditions

Lemma

Suppose wR_0x, then $\vdash S_w \to \mathrm{Con}(S_x)$.

Proof

Let h be w-OK. Then $h(l_0)R_0x$ and so $B_h \to \mathrm{Con}(S_x)$.

Lemma

Suppose wR_1x, then $\vdash S_w \to \mathrm{Con}_\omega(S_x)$.

Notation

- h is R_1-free if $l_0 = l_1$.
- h' R_0-extends h if $l_1 < l_1'$, for every $l_1 \leq i < l_1'$ it holds that $h(i)\, R_0\, h(i{+}1)$ and $h' \restriction dom(h) = h$.
- $h * x$ denotes the function $h \cup \{\langle l_1{+}1, x\rangle\}$.

Incompatibility of different Solovay formulas

Lemma

If $h \neq h'$, then $\vdash \neg(ABCD_h \wedge ABCD_{h'})$.

Proof

Case 1. There exists $i < l_1, l_1'$ such that $h(i{+}1) \neq h'(i{+}1)$.
Let i be the least with that property.

Incompatibility of different Solovay formulas

Lemma

If $h \neq h'$, then $\vdash \neg(ABCD_h \wedge ABCD_{h'})$.

Proof

Case 1. There exists $i < l_1, l_1'$ such that $h(i+1) \neq h'(i+1)$.
Let i be the least with that property.

(a) If $i < l_0, l_0'$ then
$\vdash A_h \to \exists b(\operatorname{Prf}(b, \neg S_{h(i+1)}) \wedge \forall a < b \neg \operatorname{Prf}(a, \neg S_{h'(i+1)}))$ and
$\vdash A_{h'} \to \exists b(\operatorname{Prf}(b, \neg S_{h'(i+1)}) \wedge \forall a < b \neg \operatorname{Prf}(a, \neg S_{h(i+1)}))$,
therefore A_h and $A_{h'}$ are incompatible.

Incompatibility of different Solovay formulas

Lemma

If $h \neq h'$, then $\vdash \neg(ABCD_h \wedge ABCD_{h'})$.

Proof

Case 1. There exists $i < l_1, l_1'$ such that $h(i+1) \neq h'(i+1)$.
Let i be the least with that property.

(a) If $i < l_0, l_0'$ then
$\vdash A_h \rightarrow \exists b(\,\mathrm{Prf}(b, \neg S_{h(i+1)}) \wedge \forall a < b \neg\,\mathrm{Prf}(a, \neg S_{h'(i+1)}))$ and
$\vdash A_{h'} \rightarrow \exists b(\,\mathrm{Prf}(b, \neg S_{h'(i+1)}) \wedge \forall a < b \neg\,\mathrm{Prf}(a, \neg S_{h(i+1)}))$,
therefore A_h and $A_{h'}$ are incompatible.

(b) If $l_0' \leq i < l_0$ then A_h is incompatible with $B_{h'}$.

(c) If $l_0, l_0' < i$ then $\hat{\rho}(S_h, S_{h'})$ is incompatible with $\hat{\rho}(S_{h'}, S_h)$.

Incompatibility of different Solovay formulas

Proof

Case 2 $h \subsetneq h'$

(a) If $l_1 < l'_0$ then B_h is incompatible with $A_{h'}$.

(b) If $l_1 \geq l'_0$ then $\mathrm{Con}_\omega(S_{h'(l_1+1)})$ is incompatible with $\hat{\rho}(S_{h'(l_1+1)}, S_{h'(l_1+1)})$.

Incompatibility of different Solovay formulas

Proof

Case 2 $h \subsetneq h'$

(a) If $l_1 < l'_0$ then B_h is incompatible with $A_{h'}$.

(b) If $l_1 \geq l'_0$ then $\mathrm{Con}_\omega(S_{h'(l_1+1)})$ is incompatible with $\hat{\rho}(S_{h'(l_1+1)}, S_{h'(l_1+1)})$.

Corollary

$\vdash \neg(S_x \wedge S_y)$, for every $x \neq y$.

Towards the disjunctive conditions

Lemma

Let h be R_1-free, then $\vdash A_h \to B_h \vee \bigvee_{h(l_1)R_0x} A_{h*x}$.

Lemma

Let h be R_1-free, then $\vdash A_h \to AB_h \vee \bigvee\{AB_{h'} : h' \ R_0\text{-extends } h\}$.

Towards the disjunctive conditions

Lemma

Let h be R_1-free, then $\vdash A_h \to B_h \vee \bigvee_{h(l_1)R_0 x} A_{h*x}$.

Lemma

Let h be R_1-free, then $\vdash A_h \to AB_h \vee \bigvee\{AB_{h'} : h' \ R_0\text{-extends } h\}$.

Lemma

Let h be R_1-free, then $\vdash AB_h \to ABCD_h \vee \bigvee_{h(l_1)R_1 x} ABC_{h*x}$.

Lemma

$\vdash ABC_h \to ABCD_h \vee \bigvee\{ABCD_{h'} : h' \ R_1\text{-extends } h\}$.

One Solovay condition is provable in Analysis

Lemma 1

Let h be R_1-free, then $\vdash A_h \to S_{h(l_1)} \vee \bigvee_{h(l_1) R_{\geq 0} x} S_x$.

Lemma 2

$\vdash ABC_h \to S_{h(l_1)} \vee \bigvee_{h(l_1) R_1 x} S_x$.

One Solovay condition is provable in Analysis

Lemma 1

Let h be R_1-free, then $\vdash A_h \to S_{h(l_1)} \vee \bigvee_{h(l_1) R_{\geq 0} x} S_x$.

Lemma 2

$\vdash ABC_h \to S_{h(l_1)} \vee \bigvee_{h(l_1) R_1 x} S_x$.

Corollary

$\vdash \bigvee_{x \in W \cup \{0\}} S_x$.

One Solovay condition is provable in Analysis

Lemma 1

Let h be R_1-free, then $\vdash A_h \to S_{h(l_1)} \vee \bigvee_{h(l_1)R_{\geq 0}x} S_x$.

Lemma 2

$\vdash ABC_h \to S_{h(l_1)} \vee \bigvee_{h(l_1)R_1 x} S_x$.

Corollary

$\vdash \bigvee_{x \in W \cup \{0\}} S_x$.

Proof

Let $h = \{\langle 0, 0 \rangle\}$, then trivially $\vdash A_h$ and we are done by Lemma 1.

Theorem

$\vdash S_w \to \mathrm{Prov}(\bigvee_{w\hat{R}_0 x} S_x)$ for $w \neq 0$.

Proof

Let h' be w-OK and let $h = h' \restriction \{0, \ldots, l'_0\}$.

Theorem

$\vdash S_w \to \mathrm{Prov}(\bigvee_{w\hat{R}_0 x} S_x)$ for $w \neq 0$.

Proof

Let h' be w-OK and let $h = h'{\upharpoonright}\{0, \ldots, l_0'\}$. Then one of the two hold:

- $h(l_1)\, R_1\, w$;
- $h(l_1) = w$.

And if $h(l_1)\, R_{\geq 0}\, x$, then $w\hat{R}_0 x$.

Theorem

$\vdash S_w \to \mathrm{Prov}(\bigvee_{w\hat{R}_0x} S_x)$ for $w \neq 0$.

Proof

Let h' be w-OK and let $h = h' \restriction \{0, \ldots, l'_0\}$. Then one of the two hold:

- $h(l_1)\, R_1\, w$;
- $h(l_1) = w$.

And if $h(l_1)\, R_{\geq 0}\, x$, then $w\hat{R}_0x$.

- As h is R_1-free, $\vdash A_h \to S_{h(l_1)} \vee \bigvee_{h(l_1)R_{\geq 0}x} S_x$,

Theorem

$\vdash S_w \to \mathrm{Prov}(\bigvee_{w\hat{R}_0 x} S_x)$ for $w \neq 0$.

Proof

Let h' be w-OK and let $h = h' \restriction \{0, \ldots, l'_0\}$. Then one of the two hold:

- $h(l_1)\, R_1\, w$;
- $h(l_1) = w$.

And if $h(l_1)\, R_{\geq 0}\, x$, then $w \hat{R}_0 x$.

- As h is R_1-free, $\vdash A_h \to S_{h(l_1)} \vee \bigvee_{h(l_1) R_{\geq 0} x} S_x$,
- hence $\vdash \mathrm{Prov}(A_h) \to \mathrm{Prov}\left(S_{h(l_1)} \vee \bigvee_{w\hat{R}_0 x} S_x\right)$.

Theorem

$\vdash S_w \to \mathrm{Prov}(\bigvee_{w\hat{R}_0 x} S_x)$ for $w \neq 0$.

Proof

Let h' be w-OK and let $h = h' \restriction \{0, \ldots, l'_0\}$. Then one of the two hold:

- $h(l_1)\, R_1\, w$;
- $h(l_1) = w$.

And if $h(l_1)\, R_{\geq 0}\, x$, then $w \hat{R}_0 x$.

- As h is R_1-free, $\vdash A_h \to S_{h(l_1)} \vee \bigvee_{h(l_1) R_{\geq 0} x} S_x$,
- hence $\vdash \mathrm{Prov}(A_h) \to \mathrm{Prov}\left(S_{h(l_1)} \vee \bigvee_{w\hat{R}_0 x} S_x\right)$.
- $w \neq 0$, which implies $\vdash A_h \to \mathrm{Prov}(\neg S_{h(l_1)})$

Theorem

$\vdash S_w \to \mathrm{Prov}(\bigvee_{w\hat{R}_0x} S_x)$ for $w \neq 0$.

Proof

Let h' be w-OK and let $h = h'\restriction\{0, \ldots, l'_0\}$. Then one of the two hold:

- $h(l_1)\, R_1\, w$;
- $h(l_1) = w$.

And if $h(l_1)\, R_{\geq 0}\, x$, then $w\hat{R}_0x$.

- As h is R_1-free, $\vdash A_h \to S_{h(l_1)} \vee \bigvee_{h(l_1)R_{\geq 0}x} S_x$,
- hence $\vdash \mathrm{Prov}(A_h) \to \mathrm{Prov}\left(S_{h(l_1)} \vee \bigvee_{w\hat{R}_0x} S_x\right)$.
- $w \neq 0$, which implies $\vdash A_h \to \mathrm{Prov}(\neg S_{h(l_1)})$

Since A_h is Σ^0_1, we have $\vdash A_h \to \mathrm{Prov}(A_h)$.

Theorem

$\vdash S_w \to \mathrm{Prov}_\omega(\bigvee_{wR_1x} S_x)$ for $w \neq 0$.

Proof

Let h be w-OK.

- If $l_0 = l_1$, then $\vdash A_h \to \mathrm{Prov}(\neg S_{h(l_1)})$;
- if $l_0 < l_1$, then $\vdash C_h \to \mathrm{Prov}_\omega(\neg S_{h(l_1)})$.

Therefore $\vdash ABC_h \to \mathrm{Prov}_\omega(\neg S_{h(l_1)})$.

Theorem

$\vdash S_w \to \text{Prov}_\omega(\bigvee_{wR_1x} S_x)$ for $w \neq 0$.

Proof

Let h be w-OK.

- If $l_0 = l_1$, then $\vdash A_h \to \text{Prov}(\neg S_{h(l_1)})$;
- if $l_0 < l_1$, then $\vdash C_h \to \text{Prov}_\omega(\neg S_{h(l_1)})$.

Therefore $\vdash ABC_h \to \text{Prov}_\omega(\neg S_{h(l_1)})$.

- We know that $\vdash ABC_h \to S_{h(l_1)} \vee \bigvee_{h(l_1)R_1x} S_x$;

Theorem

$\vdash S_w \to \mathrm{Prov}_\omega(\bigvee_{wR_1x} S_x)$ for $w \neq 0$.

Proof

Let h be w-*OK*.

- If $l_0 = l_1$, then $\vdash A_h \to \mathrm{Prov}(\neg S_{h(l_1)})$;
- if $l_0 < l_1$, then $\vdash C_h \to \mathrm{Prov}_\omega(\neg S_{h(l_1)})$.

Therefore $\vdash ABC_h \to \mathrm{Prov}_\omega(\neg S_{h(l_1)})$.

- We know that $\vdash ABC_h \to S_{h(l_1)} \vee \bigvee_{h(l_1)R_1x} S_x$;
- so $\vdash \mathrm{Prov}_\omega(ABC_h) \to \mathrm{Prov}_\omega\left(\bigvee_{h(l_1)R_1x} S_x\right)$.

Theorem

$\vdash S_w \to \mathrm{Prov}_\omega(\bigvee_{wR_1x} S_x)$ for $w \neq 0$.

Proof

Let h be w-OK.

- If $l_0 = l_1$, then $\vdash A_h \to \mathrm{Prov}(\neg S_{h(l_1)})$;
- if $l_0 < l_1$, then $\vdash C_h \to \mathrm{Prov}_\omega(\neg S_{h(l_1)})$.

Therefore $\vdash ABC_h \to \mathrm{Prov}_\omega(\neg S_{h(l_1)})$.

- We know that $\vdash ABC_h \to S_{h(l_1)} \vee \bigvee_{h(l_1)R_1x} S_x$;
- so $\vdash \mathrm{Prov}_\omega(ABC_h) \to \mathrm{Prov}_\omega\left(\bigvee_{h(l_1)R_1x} S_x\right)$.

Since ABC_h is Π^1_1, we have $\vdash ABC_h \to \mathrm{Prov}_\omega(ABC_h)$.

Thank You