Exercises week 4.

1. In Theorem 22 of Chapter 1 there is definitely a typo. What is this typo? Does the statement with typo still hold? If so, give a proof, if not, give a countermodel.
2. Which of the following formulas are equivalent to each other:
   - $(i)\Box(A \to B) \to (\Box A \to \Box B)$
   - $(ii)\Box A \to (\Box(A \to B) \to \Box B)$
   - $(iii)\Box B \to (\Box(A \to B) \to \Box A)$
   - $(iv)\Box B \to (\Box A \to \Box(A \to B))$
3. Is $(B \to C) \to ((A \to B) \to (A \to C))$ a tautology?
4. Consider the definition of $M, w \Vdash A$ (we will write $M, w \Vdash A$ where Boolos writes $M, w \models A$) on Page 70, Chapter 4. Remember that $\neg$, $\vee$ and $\wedge$ are all abbreviations. Proof that

$$\begin{array}{llll} (i) & w \Vdash \neg B & \text{iff} & \text{not } w \Vdash B \\ (ii) & w \Vdash A \vee B & \text{iff} & w \Vdash A \text{ or } w \Vdash B \\ (iii) & w \Vdash A \wedge B & \text{iff} & w \Vdash A \text{ and } w \Vdash B \end{array}$$

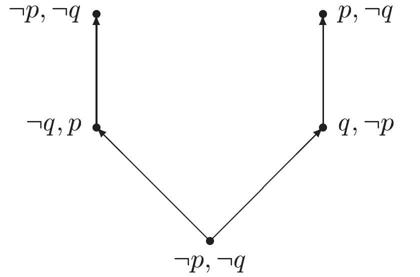
5. Explain Footnote 3 of Chapter 4.
6. In the proof of Theorem 1 of Chapter 4 we read "It follows from an easy induction on the complexity of subsentences G of F that ...". Do this induction. Is the restriction to subsentences a necessary one?
7. Read the definitions of wellfounded and converse wellfounded on Page 75. Show that the definition of wellfoundedness is equivalent to the assertion that there are no infinite descending chains. (An infinite descending chain is a chain of elements $w_0, w_1, w_2, \ldots$ such that $\ldots Rw_2Rw_1Rw_0$.)

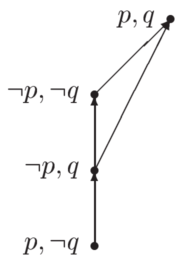
   Show that the definition of converse wellfoundedness is equivalent to the assertion that there are no infinite ascending chains. (An infinite ascending chain is a chain of elements $w_0, w_1, w_2, \ldots$ such that $w_0Rw_1Rw_2R\ldots$.)
8. Give in Exercise 2 countermodels to implications that do not hold.
9. Let σ be a substitution. by φ^σ we denote φ after the substitution σ has been performed. If for some φ and some σ we have that $\mathbf{GL} \vdash \neg(\varphi^\sigma)$, then we know that $\mathbf{GL} \nvdash \varphi$. (Why?) Provide a φ such that $\mathbf{GL} \nvdash \varphi$ and for no substitution σ, $\mathbf{GL} \vdash \neg(\varphi^\sigma)$.
10. Prove the deduction theorem for $\mathbf{GL}$. (Not in Boolos.) This is

$$\Gamma, A \vdash_{\mathbf{GL}} B \Leftrightarrow \Gamma \vdash_{\mathbf{GL}} A \to B.$$

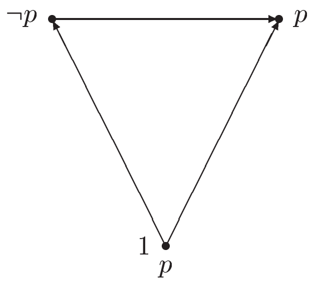
   The "$\Leftarrow$" part is very easy. For the "$\Rightarrow$" part, use induction on the length of the derivation of $\Gamma, A \vdash_{\mathbf{GL}} B$.

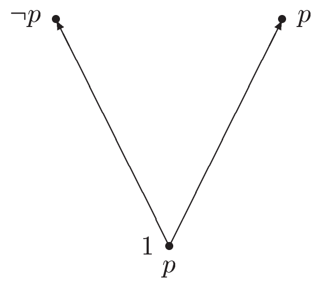
11. Fill in the gaps in Theorem 9 of Chapter 1.
12. Prove a **K4** variant of Theorem 20 of Chapter 1 of the book.
    $\mathbf{K4} \vdash \boxdot A_1 \wedge \ldots \boxdot A_n \to B \Rightarrow \mathbf{K4} \vdash \Box A_1 \wedge \ldots \Box A_n \to \Box B$.
13. Complete the proofs of Theorems 14 and 15 of Chapter 1.
14. Prove that $\mathbf{K} \vdash \Diamond(A \vee B) \leftrightarrow \Diamond A \vee \Diamond B$.
15. Prove that $\mathbf{K} \vdash \Box A \to \Box(\Box A \to A)$ and
    $\mathbf{K} \vdash \Box(\Box p \to p) \to (\Box(\Box q \to \Box p) \to \Box(\Box q \to p))$.
16. Show that
    $\mathbf{T} \vdash \Box A \to \Diamond A$ and
    $\mathbf{K4} \vdash \Box A \to (\Box\Box(A \to B) \to \Box\Box B)$.
17. Show that $\mathbf{GL} \vdash \Box(\Box^m \bot \to \Box^n \bot) \leftrightarrow \Box^{n+1}\bot$ whenever $m > n \geq 0$.
18. Determine if the following formulas are valid in the lowermost worlds of the two Kripke models below: $\Box p$, $\Box q$, $\Box p \wedge q$, $\Box\Box\bot$, $\Diamond(q \wedge \Diamond(p \wedge \neg q))$.





19. Find a formula which is true in the World 1 of the first model, but not in 1 of the second model.





20. Determine which of the following formulas are derivable in **K**:

    (a) $\Box\Box p \to \Box p$

(b) $\Box p \land \neg\Box\bot \to \neg\Box\neg p$

(c) $\Box p \land \Box\Box(p \to q) \to \Box\Box q$

(d) $\Box\Box p \land \Box\Box(p \to q) \to \Box\Box q$

Give proofs of derivable formulas and Kripke countermodels for the non-derivable ones.

21. (a) How many Kripke frames are there on a single element set? Depict them all. (b) The same question for a two-element set.

22. Assume a Kripke frame[1] has n elements and the language has m propositional variables. How many different Kripke models exist on this frame?

23. Prove the following facts by constructing appropriate Kripke countermodels:

    (a) $\mathbf{K4} \nvdash \neg p$;

    (b) $\mathbf{K4} \nvdash \Box(\Box p \to p) \to \Box p$;

    (c) $\mathbf{K4} \nvdash \neg(\Box p \to p)$;

    (d) $\mathbf{S4} \nvdash \Box(\Box(p \to \Box p) \to p) \to p$;
    (Hint: a model with just 2 nodes is sufficient.)

[1] Recall: a model is a frame together with a truth assignment of propositional variables.