Exercises week 5.

1. A At the bottom of Page 19 there is a hint at the definition of $F_v(t)$. Give this definition in full detail. (Hint: it might be an idea to work with two sequences. One sequence σ that builds up F, and another τ, that is built up almost pari passu in the following sense. Everywhere in σ where reasonable, t is substituted for v to obtain τ, except at places where v is being quantified. At these places τ will be identical to σ.)

   B On Page 43, Boolos supposes a Σ-pterm $\mathsf{sub}(t, i, x)$. Read Footnotes 5 and 9 carefully and write down a formula $\mathsf{Sub}(\mathsf{t}, \mathsf{i}, \mathsf{x}, \mathsf{y})$. Of course you are allowed to use all previously defined formulas like Finseq and the like.

2. Prove by induction on x that $\forall x \exists y\ \mathsf{Num}(x, y)$ and that $\forall x \exists! y\ \mathsf{Num}(x, y)$.

3. Should the first boldface 3 in Formula (58) of Chapter 3 really be boldface?

4. Calculate $\mathsf{Num}(3)$.

5. Calculate $\mathsf{Num}(0)$. Also calculate $\mathsf{Num}(\mathsf{Num}(0))$.

6. Calculate $\mathsf{sub}(0, 17, \ulcorner v_0 = \mathsf{S}0 \urcorner)$. Also calculate $\mathsf{su}(0, 17, \ulcorner v_0 = \mathsf{S}0 \urcorner)$.

7. A Show that if $\mathrm{PA} \vdash \mathsf{Bew}(\ulcorner A \urcorner)$, then $\mathrm{PA} \vdash A$. (Hint: If $\mathrm{PA} \vdash \varphi$, then $\mathbb{N} \models \varphi$.)

   B Show that if $\mathsf{Bew}(x)$ were a Δ-formula in PA, then PA would be inconsistent. (Hint: Use provable Σ_1-completeness.)

8. True, false or ill-defined:

   1. $\mathrm{PA} \vdash \mathsf{su}(v_j, j, \mathsf{su}(v_j, j, \ulcorner v_j = \mathsf{S}0 \urcorner)) = \ulcorner \mathsf{Num}(v_j) = \mathsf{S}0 \urcorner$
   2. $\mathrm{PA} \vdash \mathsf{su}(v_j, j, \mathsf{su}(v_j, j, \ulcorner v_j = \mathsf{S}0 \urcorner)) = \ulcorner \mathsf{Num}(\mathsf{Num}(v_j)) = \mathsf{S}0 \urcorner$
   3. $\mathrm{PA} \vdash \mathsf{su}(v_j, j, \mathsf{su}(v_j, j, \ulcorner v_j = \mathsf{S}0 \urcorner)) = \langle =, \langle \mathsf{Num}(v_j), \langle \ulcorner \mathsf{S} \urcorner, \ulcorner 0 \urcorner \rangle \rangle \rangle$
   4. None of the above options hold.

   What about $\mathsf{su}(v_j, j, \mathsf{su}(v_j, j, \mathsf{su}(v_j, j, \ulcorner v_j = \mathsf{S}0 \urcorner)))$?

9. Prove by induction on v_j that

   $$\forall v_j \forall v_i\ (v_i = v_j \to \mathsf{Bew}[v_i = v_j]) \quad (*)$$

   (In the lectures we proved $(*)$ without using induction. This yielded a shorter witness y to $\mathsf{Pf}(y, \mathsf{su}(v_j, j, \mathsf{su}(v_i, i, \ulcorner v_i = v_j \urcorner)))$. Compare this witness to the inductively defined witness in this exercise.)

10. Give the missing argument for disjunction on Page 48.

11. Do we have $\vdash \mathrm{Bew}(\ulcorner \varphi \urcorner) \to \mathrm{Bew}[\varphi]$? And do we have $\vdash \mathrm{Bew}[\varphi] \to \mathrm{Bew}(\ulcorner \varphi \urcorner)$? Provide a proof or a counterexample.

12. Show that the formula $\Box(\Box p \to q) \vee \Box(\Box q \to p)$ is valid in all linearly ordered Kripke models (more generally, if the relation R is reflexive and linear).

13. How many pairwise inequivalent formulas in one propositional variable are there (a) in classical propositional logic; (b) in **K**4.
(Answer for (b): infinitely many. Hint: iterate $\Box$. Show inequivalence by exhibiting countermodels.)

14. Find realizations $*$ and $\sharp$ such that
    - $\mathrm{PA} \vdash (\Box p)^*$
    - $\mathrm{PA} \nvdash (\Box p)^\sharp$

15. Show that $\mathbf{GL} \vdash \neg\Box\Box\bot \to (\neg\Box\neg\Box\bot \wedge \neg\Box\neg\neg\Box\bot)$. What is the arithmetical content of this formula?

16. Show that $\mathbf{GL} \vdash \Box((\Box p \to p) \to \neg\Box\Box\bot) \to \Box\Box\bot$.

17. Show that $\mathbf{K4} \vdash \Box A \to \Box(\Box A \wedge A)$

18. Show that $\mathbf{K} \vdash \Box A \to \Box(\Box\Box A \wedge \Box A \to \Box A \wedge A)$ and also that $\mathbf{K} \vdash \Box A \to \Box(\Box(\Box A \wedge A) \to \Box A \wedge A)$. Show that $\mathbf{K} \vdash \Box(\Box A \wedge A) \to \Box\Box A$. Finally show that $\mathbf{GL} \vdash \Box A \to \Box\Box A$. Prove that $\mathbf{K4} \subset \mathbf{GL}$.

19. Prove that $\mathbf{K4} \nvdash \Box(\Box A \to A) \to \Box A$. Is it possible to find a finite countermodel?

20. (a) Give an example of a **K4**-consistent formula which is not **S4**-consistent.
(b) The same question for the logics **K** and **K4**.

21. *Clonnectives* (forget about this term after this exercise (Lev says they are just called connectives)) comprise the following symbols: $\{\neg, \Box, \Diamond, \to, \wedge, \vee\}$. If a modal sentence contains n clonnectives, how many subsentences does it maximally have? Give an example where this maximum is met and give an example where this maximum is not met.

22. Show: $\Box(\Box p \to p) \to \Box p$ is true in all upwards well-founded[1], transitive Kripke models.

23. Let M be a Kripke model and $x \in M$. Show: the set $\{\phi : (M, x) \models \phi\}$ is maximal consistent.

24. Show that any maximal consistent set of formulas is closed under modus ponens.

[1] That is, there is no infinite chain $x_1 R x_2 R x_3 R \ldots$ of elements of the model.