

Well orders in the transfinite Japaridze algebra II: Turing progressions and their well-orders

David Fernández-Duque Joost J. Joosten

October 12, 2012

Abstract

This paper is a follow up to [9, 8] and studies the poly-modal provability logics GLP_Λ and natural well-founded orders therein. For each ordinal Λ one can define a propositional provability logic GLP_Λ that has for each $\alpha < \Lambda$ a modal operator $[\alpha]$ corresponding to provability and with a dual operator $\langle \alpha \rangle$ corresponding to consistency. By GLP we denote that class size logic that has a modality for each ordinal.

Worms are iterated consistency statements of the form $\langle \alpha_1 \rangle \dots \langle \alpha_n \rangle \top$. We denote the class of all worms by S and by S_α we denote the class of worms all of whose occurring modalities are at least α .

Worms A, B in S_α can be naturally ordered by defining $A <_\alpha B :\Leftrightarrow \text{GLP} \vdash B \rightarrow \langle \alpha \rangle A$. The ordered structures $\langle S_\alpha, <_\alpha \rangle$ have been extensively studied ([4, 1, 9]) and it is known that, modulo provable equivalence, they define well orders and constitute alternative ordinal notation systems.

In particular, for each worm $A \in S_\alpha$ the set $\{B \in S_\alpha \mid B <_\alpha A\}$ is, modulo provable equivalence, a well-order with an order-type we shall denote $o_\alpha(A)$. In this paper we see how a small change in the definition of these sets makes a tremendous difference: the sets $\{B \in S \mid B <_\alpha A\}$ are well-founded but exhibit infinite anti-chains for $\alpha > 0$.

We will denote the supremum of order types of chains in $\{B \in S \mid B <_\alpha A\}$ by $\Omega_\alpha(A)$. In this paper we shall fully characterize which sequences of ordinals can be attained as $\langle \Omega_\xi(A) \rangle_{\xi \in \text{Ord}}$ for some worm A and dwell on the link of these sequences to Turing progressions and to modal semantics for the closed fragment of GLP_Λ .

We give both a local and a global characterization of such sequences. The global characterization is given in terms of so-called *cohyperations* of ordinal functions. Cohyperations are defined as an infinite iterate of particular ordinal functions.

1 Introduction

The provability logic of an arithmetic theory T is a modal logic where the \square modality is interpreted as the formalization of “provable in T ”. The structural

propositional behavior of formalized provability in sound r.e. theories is characterized by Solovay’s theorem [15] and is known to be the modal logic **GL** that we shall introduce below.

It is known that provability logics are very stable in that *any* sound r.e. theory that extends some rather weak arithmetic theory as $I\Delta_0 + \text{exp}$ has the same provability logic **GL**. And, as a matter of fact, one can also weaken the assumption of recursively enumerable axiomatizability. In particular it is known that **GL** is also the logic of provability when interpreting the \Box operator as “provable with n applications of the omega-rule” or “provable in T together with all true Π_n -formulas”, etc.

Japaridze ([12]) introduced a logic GLP_ω (details follow below) that has for each natural number n a modality $[n]$ where we interpret $[n]$ as “provable by n applications of the ω -rule”. He showed this logic GLP_ω to be arithmetically sound and complete for this interpretation. Ignatiev then showed in [11] that this completeness result actually holds for a wide range of arithmetical readings of $[n]$.

In particular, we still have completeness of GLP_ω when reading $[n]$ as a natural formalization of “provable in EA together with all true Π_n^0 sentences”. We shall see that under this reading, the logic GLP_ω is closely related to Turing progressions (also defined below).

Interest in the logic GLP_ω and related systems recently revived when Beklemishev applied GLP_ω to perform a Π_1^0 -ordinal analysis for Peano arithmetic (PA) and related systems ([2]).

Moreover, it turned out that GLP_ω and fragments poses very interesting properties. One can debate over the notion of natural, but arguably it is the first natural example of a logic that is not Kripke complete but that is complete with respect to their natural class of topological spaces [?]. However, if one restricts oneself to natural ordinal spaces with their corresponding canonical topologies then the question of completeness becomes dependent on set-theoretical assumptions which are themselves independent of ZFC ([?, ?]).

The ordinal analysis that Beklemishev performed for PA and its kin was actually carried out almost entirely within the closed fragment GLP_ω^0 of GLP_ω , that is, those theorems of GLP_ω that do not contain propositional variables but rather are built up from \perp , \top and the modal and Boolean connectives. Particular terms –so called *worms*– within this fragment constitute an alternative ordinal notation system for ordinals below ε_0 .

In order to obtain an ordinal notation system based on worms that goes beyond ε_0 , Beklemishev considered in [4] the logics GLP_Λ with $\Lambda > \omega$. These logics are like GLP_ω with the sole exception that they now contain a modality $[\alpha]$ for each $\alpha < \Lambda$ together with their corresponding axioms and rules. Beklemishev also introduced a class-size logic GLP that contains a modality $[\alpha]$ for each ordinal α .

In [4], Beklemishev also established a correspondence between the ordinal notation system based on worms and the more familiar one based on so-called Veblen normal forms. This relation was studied in more detail in [9] where in particular the authors worked with so-called *hyperations* instead of Veblen

functions.

Hyperations are transfinite iterations of normal ordinal functions which can be seen as a natural refinement of the Veblen functions in particular, and more in general of any Veblen progression of normal ordinal functions. The theory of hyperations is inspired by problems that arose in the study of GLP but is studied and developed on an independent footing by the authors in [7].

In the current paper we study natural and important generalizations of the orderings on worms that were studied in both [4] and [9]. Parts of the results presented in this paper have been presented in [8]. In the current paper, these results are presented with more detail. Moreover, we present an important improvement that is based on the techniques of so-called *cohyperations*. In order to sketch an outline of this paper we first need to formally introduce the logics GLP_Λ , their closed fragments and the worms that dwell therein.

1.1 The logics GLP_Λ

The language of GLP_Λ is that of propositional modal logic that contains for each $\alpha < \Lambda$ a unary modal operator $[\alpha]$. In the definition below the α and β range over ordinals and the ψ and χ over formulas in the language of GLP_Λ .

Definition 1.1. *For Λ an ordinal, the logic GLP_Λ is the propositional normal modal logic that has for each $\alpha < \Lambda$ a modality $[\alpha]$ and is axiomatized by the following schemata:*

$$\begin{aligned} &[\alpha](\chi \rightarrow \psi) \rightarrow ([\alpha]\chi \rightarrow [\alpha]\psi), \\ &[\alpha]([\alpha]\chi \rightarrow \chi) \rightarrow [\alpha]\chi, \\ &\langle \alpha \rangle \psi \rightarrow [\beta] \langle \alpha \rangle \psi && \text{for } \alpha < \beta, \\ &[\alpha]\psi \rightarrow [\beta]\psi && \text{for } \alpha \leq \beta. \end{aligned}$$

The rules of inference are *Modus Ponens* and *necessitation* for each modality: $\frac{\psi}{[\alpha]\psi}$. By GLP we denote the class-size logic that has a modality $[\alpha]$ for each ordinal α and all the corresponding axioms and rules.

It is good to recall that from Löb's axiom $[\alpha]([\alpha]\chi \rightarrow \chi) \rightarrow [\alpha]\chi$ one can easily derive transitivity, that is,

$$[\alpha]\chi \rightarrow [\alpha][\alpha]\chi,$$

and we shall use this freely in our reasoning. The classical Gödel-Löb provability logic **GL** is denoted by GLP_1 .

1.2 Worms and the closed fragment of GLP

A closed formula in the language of GLP is simply a formula without propositional variables. In other words, closed formulas are generated by just \top and the Boolean and modal operators.

The closed fragment of GLP is just the class of closed formulas provable in GLP and is denoted by GLP^0 . Within this closed fragment and the corresponding

algebra, there is a particular class of privileged inhabitants/terms which are called *worms*.

Definition 1.2 (Worms, S , S_α). *By S we denote the set of worms of GLP which is inductively defined as $\top \in S$ and $A \in S \Rightarrow \langle \alpha \rangle A \in S$. Similarly, we inductively define for each ordinal α the set of worms S_α where all ordinals are at least α as $\top \in S_\alpha$ and $A \in S_\alpha \wedge \beta \geq \alpha \Rightarrow \langle \beta \rangle A \in S$.*

Both the closed fragment of GLP and the set of worms have been studied in [4] and [1]. Worms can be conceived as the backbone of GLP^0 and obtain their name from the heroic worm-battle, a variant of the Hydra battle (see [5]).

We shall identify a worm A in the obvious way with $\iota(A)$, the string of ordinals in the consistency statements that is involved in A : $\iota(\top) = \lambda$ and $\iota(\langle \alpha \rangle A) = \alpha * \iota(A)$. In this paper λ will denote the empty string.

Apart from identifying a worm with its corresponding string of ordinals we shall use any hybrid combination in between at times. For example, we might equally well write 10ω , as $\langle 1 \rangle 0\omega$, or $\langle 1 \rangle \langle 0 \rangle \langle \omega \rangle \top$.

The following lemma follows easily from the axioms of GLP and shall be used repeatedly without explicit mention in the remainder of this paper.

Lemma 1.3.

1. For a GLP formula ϕ and a worm B , if $\beta < \alpha$, then

$$\text{GLP} \vdash (\langle \alpha \rangle \phi \wedge \langle \beta \rangle B) \leftrightarrow \langle \alpha \rangle (\phi \wedge \langle \beta \rangle B);$$
2. If $A \in S_{\alpha+1}$, then $\text{GLP} \vdash A \wedge \langle \alpha \rangle B \leftrightarrow A\alpha B$;
3. If $A, B \in S_\alpha$ and $\text{GLP} \vdash A \leftrightarrow B$, then

$$\text{GLP} \vdash A\alpha C \leftrightarrow B\alpha C.$$

Proof. The \rightarrow direction of the first item follows from the axiom $\langle \beta \rangle B \rightarrow [\alpha] \langle \beta \rangle B$. For the other direction we observe that $\langle \alpha \rangle \langle \beta \rangle B \rightarrow \langle \beta \rangle B$ in virtue of axiom $\langle \alpha \rangle \langle \beta \rangle B \rightarrow \langle \beta \rangle \langle \beta \rangle B$ and transitivity of $[\beta]$. The other two items follow directly from the first. \square

1.3 Plan of the paper

After the introduction, in Section 3 we will revisit some standard notions from ordinal arithmetic that are needed throughout the rest of the paper.

In Section 4 we describe the linear orders $<_\alpha$ on S_α defined as $A <_\alpha B :\Leftrightarrow \text{GLP} \vdash B \rightarrow \langle \alpha \rangle A$. The function o will map a worm to the order type of the set $\{B \in S \mid B <_0 A\}$. We resume a calculus for computing o as presented in [9]. An important ingredient in this calculus is the function e^α which is defined as the function that enumerates $o(S_\alpha)$. The functions e^α can be seen as a transfinite iterate that we call *hyperation*.

Next, in Section 5 we study the order $<_\alpha$ on S in general and not only on S_α . In this case $<_\alpha$ no longer linearly orders S but rather defines a well-founded relation. By $\Omega_\alpha(A)$ we will denote the supremum of order-types of linear orders

that reside in $\{B \in S \mid B <_\alpha A\}$. We shall see how the study of Ω_α 's can be recursively reduced to the study of o_ξ 's. Most of the results presented here and in the next section appeared also in [8].

In Section 6 we shall study the sequences $\langle \Omega_\alpha(A) \rangle_{\alpha \in \mathbf{O}_n}$ for worms A and give a full characterization of these sequences.

We shall see that these sequences are important for two reasons. Firstly, in Section 5 we see that they provide us information (lower-bounds, one could say) of what a modal model for the closed fragment of GLP should look like. In Section 10 we shall see that the theory $T + A$ can exactly be characterized in terms of its Turing progression approximations by $\langle \Omega_\alpha(A) \rangle_{\alpha \in \mathbf{O}_n}$.

The first characterization of these omega sequences that we give is of local nature. In particular, there is a lemma that determines the nature of the omega-sequences at successor ordinals, and a different lemma for limit ordinals.

In Section 7 we take these two lemmata under the loupe and isolate a common feature. To smoothly express this common feature we would need a uniform way to obtain left-inverses to hyperations: which is given by the theory of what we call *cohyperations*.

In Section 8 we summarize results from the theory of hyperations and cohyperations as presented in [7]. An important theorem is obtained that characterizes so-called *hyperlogarithms* which are essential in the next section. Hyperations and cohyperations were introduced by the authors in order to give a smooth global presentation of the omega sequences.

Finally, in Section 9 we set the cohyperations at work to obtain a global characterization of the omega sequences.

1.4 Notation

We reserve lower-case Greek letters $\alpha, \beta, \gamma, \dots, \xi \dots$ for variables ranging over ordinals. Worms will be denoted by upper case latin letters A, B, C, \dots . The Greek lower-case letters ϕ, ψ, χ, \dots will denote formulas. However, φ shall be reserved for the Veblen enumeration function and variants thereof. Likewise, we reserve ω to denote the first infinite ordinal.

2 Turing progressions and modal logic

The logics **GL** and GLP_ω turn out to be very well suited to talk about Turing progressions. Let us recall the definition of Turing progressions as introduced by Turing in his seminal paper [16].

Gödel's Second Incompleteness Theorem tells us that any sound recursive theory that is strong enough to code syntax will not prove its own consistency. Thus, adding $\text{Con}(T)$ to such a theory T will yield a strictly stronger theory. Turing took up this idea to consider recursive ordinal progressions of some

recursive sound base theory T :

$$\begin{aligned} T_0 &:= T; \\ T_{\alpha+1} &:= T_\alpha + \text{Con}(T_\alpha); \\ T_\lambda &:= \bigcup_{\alpha < \lambda} T_\alpha \quad \text{for limit } \lambda. \end{aligned}$$

Poly-modal provability logics turn out to be suitably well equipped to talk about Turing progressions. When talking about closed formulas of GLP we shall often not distinguish a modal formula from its arithmetical interpretation.

Finite Turing progressions are definable in \mathbf{GL} as T_n is provably equivalent to $T + \diamond_T^n \top$. Transfinite progressions are not expressible in the modal language with just one modal operator. However, using stronger provability predicates provides a way out (see [3]). In particular, the following proposition tells us how to approximate the ω 'th Turing progression. For this and the following proposition there are some technical side-conditions on the theory T that shall be specified in Section 10. In the current section, we are mainly interested in seeing the link between Turing progressions and polymodal provability logics.

Proposition 2.1. *$T + \langle n+1 \rangle_T \top$ is a Π_{n+1} conservative extension of $T + \{\langle n \rangle_T^k \top \mid k \in \omega\}$.*

More in general we have the following proposition ([3]):

Proposition 2.2. *For each ordinal $\alpha < \epsilon_0$ there is some GLP_ω -worm A such that $T + A$ is Π_1 equivalent to T_α .*

To get generalizations of this lemma beyond ϵ_0 one should consider more than ω modalities. Before doing so, in the next section we first provide some more background on the ordinals that we shall need later on in this paper. In the final section, Section 10, we shall see how the omega sequences can be interpreted in terms of Turing progressions.

3 Ordinal arithmetic

In this section we shall briefly state without proof the main properties of ordinals that we need in the remainder of this paper. For further definitions and detailed proofs, we refer the reader to [?]. Ordinals are canonical representatives for well-orders. The first infinite ordinal is as always denoted by ω .

Most operations on natural numbers can be extended to ordinal numbers, like addition, multiplication and exponentiation (see [?]).

Lemma 3.1.

1. $\forall \zeta < \xi \exists! \eta \zeta + \eta = \xi$
(We will denote this unique η by $-\zeta + \xi$),
2. $\forall \eta > 0 \exists \alpha \exists! \beta \eta = \alpha + \omega^\beta$
(We will denote this unique β by $\ell\eta$),

3. $\forall \eta > 0 \exists! \alpha, \beta \eta = \omega^\alpha + \beta$ such that $\beta < \omega^\alpha + \beta$.

One of the most useful ways to represent ordinals is through their Cantor Normal Forms (CNFs):

Theorem 3.2 (Cantor Normal Form Theorem).

For each ordinal α there are unique ordinals $\alpha_1 \geq \dots \geq \alpha_n$ such that

$$\alpha = \omega^{\alpha_1} + \dots + \omega^{\alpha_n}.$$

We call a function f *increasing* if $\alpha < \beta$ implies $f(\alpha) < f(\beta)$. An ordinal function is called *continuous* if $\bigcup_{\zeta < \xi} f(\zeta) = f(\xi)$ for all limit ordinals ξ . Functions which are both increasing and continuous are called *normal*.

It is not hard to see that each normal function has an unbounded set of fixpoints. For example the first fixpoint of the function $\varphi_0 : x \mapsto \omega^x$ is

$$\sup\{\omega, \omega^\omega, \omega^{\omega^\omega}, \dots\}$$

and is denoted ε_0 . Clearly for these fixpoints, CNFs give little information as, for example, $\varepsilon_0 = \omega^{\varepsilon_0}$. Therefore, we shall need notations and normal forms that are slightly more informative and which are based on functions that enumerate the fixpoints of normal functions: Veblen Normal Forms (VNFs).

In his seminal paper [17], Veblen considered for each normal function f its derivative f' that enumerates the fixpoints of f . If f is a normal function, then the image of f –which we shall denote by F – is a closed (under taking suprema) unbounded set. Likewise the function that enumerates a closed unbounded set is continuous. For f a normal function, we define F' to be the image of f' and we extend this transfinitely by setting

$$\begin{aligned} F_{\alpha+1} &:= (F_\alpha)'; \\ F_\lambda &:= \bigcap_{\alpha < \lambda} F_\alpha \quad \text{for limit } \lambda, \end{aligned}$$

then taking f_λ to be the function that enumerates F_λ .

By taking $\Phi_0 := \{\omega^\alpha \mid \alpha \in \text{On}\}$ one obtains Veblen's original hierarchy and the φ_α denote the corresponding enumeration functions of the classes Φ_α .

Beklemishev noted in [4] that in the setting of GLP it is desirable to have $1 \notin X_0$. Thus he considered the progression that started with $\Phi_0^B := \{\omega^{1+\alpha} \mid \alpha \in \text{On}\}$. We denote the corresponding enumeration functions by $\hat{\varphi}_\alpha$.

In [7] and in this paper the authors realized that, moreover it is desirable to have 0 in the initial set, whence we departed from

$$E_0 = \{0\} \cup \{\omega^{1+\alpha} \mid \alpha \in \text{On}\}.$$

We shall denote the corresponding enumeration functions by e_α . In general, if f is some normal function, we shall denote by f_α the Veblen progression based on $f_0 = f$. Note that, if $\alpha < \beta$, we have that $f_\beta(\gamma)$ is always a fixpoint of f_α , i.e., $f_\beta = f_\alpha \circ f_\beta$.

One readily observes that

$$\begin{aligned} e_\alpha(0) &= 0 && \text{for all } \alpha; \\ e_0(1+\beta) &= \varphi_0(1+\beta) = \hat{\varphi}_0(\beta) && \text{for all } \beta; \\ e_{1+\alpha}(1+\beta) &= \varphi_{1+\alpha}(\beta) = \hat{\varphi}_{1+\alpha}(\beta) && \text{for all } \alpha, \beta. \end{aligned}$$

Many times, we can write an ordinal ω^α in more than one way as $\varphi_\xi(\eta)$. However, if we require that $\eta < \varphi_\xi(\eta)$, then both ξ and η are uniquely determined. In other words

$$\forall \alpha \exists! \eta, \xi [\omega^\alpha = \varphi_\xi(\eta) \wedge \eta < \varphi_\xi(\eta)].$$

Combining this fact with the CNF Theorem one obtains *Veblen Normal Forms* for ordinals.

Theorem 3.3 (Veblen Normal Form Theorem). *For all α there exist unique $\alpha_1, \beta_1, \dots, \alpha_n, \beta_n$ ($n \geq 0$) such that*

1. $\alpha = \varphi_{\alpha_1}(\beta_1) + \dots + \varphi_{\alpha_n}(\beta_n)$,
2. $\varphi_{\alpha_i}(\beta_i) \geq \varphi_{\alpha_{i+1}}(\beta_{i+1})$ for $i < n$,
3. $\beta_i < \varphi_{\alpha_i}(\beta_i)$ for $i \leq n$.

Note that $\alpha_i \geq \alpha_{i+1}$ does not in general hold in the VNF of α . For example,

$$\omega^{\varepsilon_0+1} + \varepsilon_0 = \varphi_0(\varepsilon_0 + 1) + \varphi_1(0) = \varphi_0(\varphi_{\varphi_0(0)}(0) + \varphi_0(0)) + \varphi_{\varphi_0(0)}(0).$$

4 Linear orders on the Japaridze algebra

In this section we shall introduce linear orders on worms, an important theme in our paper.

4.1 The orderings $<_\alpha$

It is known that the class of worms is modulo provable equivalence linearly ordered by consistency strength. That is, two worms are either equivalent or one of the two implies the consistency (0-consistency that is) of the other.

Definition 4.1 ($<, <_\alpha, o, o_\alpha$). *We define a relation $<_\alpha$ on $S_\alpha \times S_\alpha$ by*

$$A <_\alpha B \text{ :} \Leftrightarrow \text{GLP} \vdash B \rightarrow \langle \alpha \rangle A \quad (\text{with } A, B \in S_\alpha).$$

For $A \in S_\alpha$ we denote by $o_\alpha(A)$ the order type of $\{B \in S_\alpha \mid B <_\alpha A\}$. More precisely, for $A \in S_\alpha$ we define inductively

$$o_\alpha(A) = \sup \{o_\alpha(B) + 1 : B \in S_\alpha \ \& \ B <_\alpha A\},$$

where $\sup \emptyset = 0$.

When X is a set or class we shall denote by $o_\alpha(X)$ the image of X under o_α .

Instead of $<_0$ and o_0 we shall write $<$ and o , respectively. In [9] we described Japaridze algebras and how these algebras are the environments where one most naturally considers our orderings.

As mentioned before, the relations $<_0$ defines total ordering on S_0 modulo provable equivalence. In the following subsection we see how we can choose natural representatives from the equivalences classes by switching to what we call *Beklemishev Normal Forms*.

4.2 A well-order on Beklemshev Normal Forms

BNFs are a subclass of S on which $<_0$ does define a linear order as was shown in [1, 4]. In those papers it was also shown that each worm is equivalent to a unique worm in BNF and that this BNF can be found effectively for recursive well-orders. Moreover, if $A \in S_\alpha$, then its equivalent in BNF is also in S_α .

In this section we shall provide a calculus to compute o_α . Note that it is not at all obvious that o_α is defined everywhere, but this turns out to be the case.

Definition 4.2 (Beklemishev Normal Form). *A worm $A \in S$ is in BNF (Beklemishev Normal Form) iff*

1. $A = \lambda$ *or,*
2. *A is of the form $A_k \alpha \dots \alpha A_1$ with $\alpha = \min(A)$, $k \geq 1$ and $A_i \in S_{\alpha+1}$ such that each A_i is in BNF and moreover $A_{i+1} \leq_{\alpha+1} A_i$ for each $i < k$.*

We shall write \mathcal{B} for BNF and \mathcal{B}_α for $\text{BNF} \cap S_\alpha$.

Lemma 4.3. *Each worm of the form α^n , i.e., $\overbrace{\langle \alpha \rangle \dots \langle \alpha \rangle}^{n \text{ times}} \top$, is in BNF.*

Proof. This is immediate if we conceive α^n as $\lambda \alpha \lambda \dots \lambda \alpha \lambda$. □

As announced before, the BNFs form a class of natural representatives for formulas without variables with respect to o :

Lemma 4.4. *The map $o : (\mathcal{B}, <_0) \rightarrow (\text{Ord}, <)$ defines an isomorphism.*

4.3 A calculus for o

In this subsection we state a calculus for computing o and o_α . Proofs and details of the calculus presented here can be found in [9]. We first need a syntactical operation that promotes or demotes worms in terms of consistency strength.

Definition 4.5 ($\alpha \uparrow$ and $\alpha \downarrow$). *Let A be a worm and α an ordinal. By $\alpha \uparrow A$ we denote the worm that is obtained by simultaneously substituting each β that occurs in A by $\alpha + \beta$.*

Likewise, if $A \in S_\alpha$ we denote by $\alpha \downarrow A$ the worm that is obtained by replacing simultaneously each β in A by $-\alpha + \beta$.

Note that by Lemma 3.1, the operation $\alpha \downarrow$ is well-defined on S_α . The next lemma enumerates some noteworthy properties of these promoting and demoting operations.

Lemma 4.6. *For α, β, γ ordinals and worms A, B we have:*

1. $\alpha \uparrow \beta < \alpha \uparrow \gamma \Leftrightarrow \beta < \gamma$,
2. $\alpha \uparrow \beta \geq \beta$,
3. $\alpha \uparrow (\beta \uparrow A) = (\alpha + \beta) \uparrow A$,
4. $\alpha \downarrow (\beta \uparrow A) = (-\alpha + \beta) \uparrow A$, provided $\alpha \leq \beta$,
5. $\alpha \downarrow (\beta \downarrow A) = (\beta + \alpha) \downarrow A$, provided $A \in S_{\beta+\alpha}$,
6. $\alpha \uparrow ((\beta + \alpha) \downarrow A) = \beta \downarrow A$ for $A \in S_{\beta+\alpha}$,
7. $A <_\alpha B \Leftrightarrow A < B$ for $A, B \in S_\alpha$,
8. $A <_\xi B \Leftrightarrow \alpha \uparrow A <_{\alpha+\xi} \alpha \uparrow B$.

Proof. Proofs for the non-trivial items can be found in [9]. The new items can be easily proved by elementary methods. For example, for Item 5 we see by definition, that for large enough γ we have

$$\alpha + (\alpha \downarrow (\beta \downarrow \gamma)) = \beta \downarrow \gamma. \quad (1)$$

Likewise we have $\beta + (\beta \downarrow \gamma) = \gamma$. If we now substitute (1) in the latter we obtain

$$\beta + (\alpha + (\alpha \downarrow (\beta \downarrow \gamma))) = (\beta + \alpha) + (\alpha \downarrow (\beta \downarrow \gamma)) = \gamma.$$

Whence, by definition we see that $\alpha \downarrow (\beta \downarrow \gamma) = (\beta + \alpha) \downarrow \gamma$. \square

Moreover, in [9] it is proven that $\alpha \uparrow$ is a well-behaved map with nice properties. In particular, $\alpha \uparrow$ can also be viewed as an isomorphism:

Lemma 4.7. *The map $\alpha \uparrow$ is an isomorphism between $(S, <)$ and $(S_\alpha, <_\alpha)$.*

In [9] we introduced the functions e^α that we call *hyperexponentials*.

Definition 4.8. *For ordinals α and β , the values $e^\alpha(\beta)$ are determined by the following recursion.*

1. $e^{\alpha 0} = 0$ for all $\alpha \in \text{Ord}$;
2. $e^1 = e$ where e enumerates the set $\{0\} \cup \{\omega^{1+\alpha} \mid \alpha \in \text{Ord}\}$;
3. $e^{\alpha+\beta} = e^\alpha e^\beta$;
4. $e^\alpha(\lambda) = \cup_{\beta < \lambda} e^\alpha(\beta)$ for limit ordinals λ ;
5. $e^\lambda(\beta + 1) = \cup_{\lambda' < \lambda} e^{\lambda'}(e^\lambda(\beta) + 1)$ for λ an additively indecomposable limit ordinal.

Based on these hyperexponential functions e^α we can formulate an elegant calculus to compute the values of $o_\alpha(A)$:

Theorem 4.9.

1. $o(0^n) = n$;
2. If $A = A_n 0 \dots A_1 \in \mathcal{B}$ and $A_1 \in \mathcal{B}_1$ is not empty, then $o(A) = \omega^{o(1 \downarrow A_1)} + \dots + \omega^{o(1 \downarrow A_n)}$, where for $n = 1$ we denote by $A_n 0 \dots A_1$ simply A_1 ;
3. $o(\xi \uparrow A) = e^\xi o(A)$,
4. $o_\xi(A) = o(\xi \downarrow A)$ for $A \in S_\xi$.

Note that the last item of this theorem is not needed to compute o . It merely tells us how to reduce o_α to o . The e^α functions can be related to the more familiar Veblen progressions.

Lemma 4.10. $e^{\omega^\alpha} = e_\alpha$.

Moreover, we note that Lemma 4.10 together with Theorem 4.8.3 yields a reduction of computing e^α to the better known Veblen-like functions e_α . For if $\alpha = \omega^{\alpha_1} + \dots + \omega^{\alpha_n}$, then

$$e^\alpha = e_{\alpha_1} \circ \dots \circ e_{\alpha_n}.$$

5 Well-founded orders on worms

In this section we consider the ordering $<_\alpha$ on the full $S \times S$ rather than on $S_\alpha \times S_\alpha$. We shall see that the resulting order is still well-founded but no longer total. Most of the results presented here and in the next section –with the exception of Subsection 5.4– were also presented in [8].

5.1 Well-founded orders

In Section 4 we presented the well-orders $<_\alpha$ on S_α . We can also consider the ordering $<_\alpha$ on the full class S . As we shall see, $<_\alpha$ is no longer linear on S . However, it is still well-founded. Anticipating this, we can define $\Omega_\alpha(A)$, the generalized $<_\alpha$ order-type of a worm A .

Definition 5.1. Given an ordinal ξ and a worm A , we define a new ordinal $\Omega_\xi(A)$ inductively on $<_\xi$ by

$$\Omega_\xi(A) = \sup_{B <_\xi A} (\Omega_\xi(B) + 1).$$

With this, we can assign to each worm A a sequence of order-types.

We will use the notation $\vec{\Omega}(A)$ for the sequence $\langle \Omega_\xi(A) \rangle_{\xi \in \text{On}}$; that is,

$$\vec{\Omega}(A) := (\Omega_0(A), \Omega_1(A), \dots, \Omega_\omega(A), \Omega_{\omega+1}(A) \dots).$$

We shall refer to these sequences as *Omega-sequences*.

5.2 Omega-sequences and modal semantics

Each worm A is known to be consistent with GLP, hence should be satisfied in an exact model for its closed fragment, if it exists; that is, a model on which only the theorems of GLP^0 are valid.

Suppose \mathcal{M} were such a model. Each modality $\langle \xi \rangle$ will be represented in \mathcal{M} by some relation \prec_ξ in that

$$\mathcal{M}, w \Vdash \langle \xi \rangle \phi \Leftrightarrow \exists w' (w' \prec_\xi w \wedge \mathcal{M}, w' \Vdash \phi).$$

As $[\xi]$ satisfies Löb's axiom, we know that each \prec_ξ is transitive and well-founded. Consequently, we can assign to each world w a sequence of ordinals

$$\vec{w} := (w_0, w_1, \dots, w_\omega, w_{\omega+1} \dots),$$

where w_ζ corresponds to the supremum of order-types of \prec_ζ -chains below w . If $\mathcal{M}, w \Vdash A$, then necessarily $w_\xi \geq \Omega_\xi(A)$ for each ξ . A systematic study of $\vec{\Omega}(A)$ will thus also reveal information about models for GLP^0 .

No such models were known, but in [6] the authors define a universal class-size model for GLP^0 . The worlds in that model closely reflect the $\Omega_\xi(A)$ sequences as defined here. In particular, it turns out that the necessary condition that if $\mathcal{M}, w \Vdash A$, then $w_\xi \geq \Omega_\xi(A)$ for each ξ is actually also sufficient.

In Section 6 we shall characterize the sequences $\Omega_\xi(A)$ for given ξ and A . In the next subsection we shall see how questions about Ω_ξ can be recursively reduced to questions about o_ζ .

5.3 Reducing Ω_ξ to o_ζ

In Lemma 5.4 below we shall see how questions about Ω_ξ can be recursively reduced to questions about o_ζ . For this reduction we need the syntactical definitions of *head* and *remainder*.

Definition 5.2. *Let A be a worm. By $h_\xi(A)$ we denote the ξ -head of A . Recursively: $h_\xi(\lambda) = \lambda$, $h_\xi(\zeta * A) = \zeta * h_\xi(A)$ if $\zeta \geq \xi$ and $h_\xi(\zeta * A) = \lambda$ if $\zeta < \xi$.*

*Likewise, by $r_\xi(A)$ we denote the ξ -remainder of A : $r_\xi(\lambda) = \lambda$, $r_\xi(\zeta * A) = r_\xi(A)$ if $\zeta \geq \xi$ and $r_\xi(\zeta * A) = \zeta * A$ if $\zeta < \xi$.*

In words, $h_\xi(A)$ corresponds to the largest initial part (reading from left to right) of A such that all symbols in $h_\xi(A)$ are at least ξ and $r_\xi(A)$ is that part of A that remains when removing its ξ -head. We thus have $A = h_\xi(A) * r_\xi(A)$ for all ξ and A .

Observe that

$$\text{GLP} \vdash h_\xi(A) * r_\xi(A) \leftrightarrow h_\xi(A) \wedge r_\xi(A), \quad (2)$$

as the first symbol of $r_\xi(A)$ is less than ξ and $h_\xi(A) \in S_\xi$ (see Lemma 1.3). Moreover, for each ξ and each A we have that $h_\xi(A)$ is in normal form whenever A is:

Lemma 5.3. *If $A \in \text{BNF}$, then also $h_\zeta(A) \in \text{BNF}$ and $r_\zeta(A) \in \text{BNF}$.*

Proof. We prove here the $h_\zeta(A)$ case. For $A = \lambda$ this is clear. Thus, let the symbols in A be enumerated in increasing order by ξ_0, \dots, ξ_n . By an easy induction on n we see that each $h_{\xi_i}(A) \in \text{BNF}$. If $\xi_n > \zeta \notin A$, then $h_\zeta(A) = h_{\min\{\xi_i | \xi_i > \zeta\}}(A)$. If $\zeta > \xi_n$, then $h_\zeta(A) = \lambda$ which is in BNF . \square

Lemma 5.4. *Let A and B be worms. We have that*

$$(A \rightarrow \langle \xi \rangle B) \Leftrightarrow [(h_\xi(A) \rightarrow \langle \xi \rangle h_\xi(B)) \wedge (A \rightarrow r_\xi(B))].$$

Proof. “ \Rightarrow ” By 2, $B \leftrightarrow h_\xi(B) \wedge r_\xi(B)$ whence $A \rightarrow r_\xi(B)$ as

$$\begin{aligned} A &\rightarrow \langle \xi \rangle B \\ &\rightarrow \langle \xi \rangle (h_\xi(B) \wedge r_\xi(B)) \quad \text{by Lemma 1.3.2} \\ &\rightarrow r_\xi(B) \wedge \langle \xi \rangle h_\xi(B) \\ &\rightarrow r_\xi(B). \end{aligned}$$

Likewise $A \leftrightarrow h_\xi(A) \wedge r_\xi(A)$. As $h_\xi(A), h_\xi(B) \in S_\xi$ we know that either

- $h_\xi(A) = h_\xi(B)$,
- $h_\xi(B) \rightarrow \langle \xi \rangle h_\xi(A)$ or,
- $h_\xi(A) \rightarrow \langle \xi \rangle h_\xi(B)$.

By assumption $A \rightarrow \langle \xi \rangle B$ whence $A \rightarrow \langle \xi \rangle h_\xi(B) \wedge r_\xi(B)$.

Suppose now $h_\xi(A) = h_\xi(B)$. Then,

$$h_\xi(A) \wedge r_\xi(A) \rightarrow \langle \xi \rangle h_\xi(A) \wedge r_\xi(A)$$

whence also

$$h_\xi(A) \wedge r_\xi(A) \rightarrow \langle \xi \rangle (h_\xi(A) \wedge r_\xi(A)).$$

The latter is equivalent to $A \rightarrow \langle \xi \rangle A$ which contradicts the irreflexivity of $<_\xi$.

By a similar argument, the assumption that $h_\xi(B) \rightarrow \langle \xi \rangle h_\xi(A)$ contradicts the irreflexivity of $<_\xi$ and we conclude that $h_\xi(A) \rightarrow \langle \xi \rangle h_\xi(B)$.

“ \Leftarrow ” This is the easier direction.

$$\begin{aligned} A &\leftrightarrow h_\xi(A) \wedge r_\xi(A) \\ &\rightarrow \langle \xi \rangle h_\xi(B) \wedge r_\xi(B) \\ &\rightarrow \langle \xi \rangle (h_\xi(B) \wedge r_\xi(B)) \\ &\rightarrow \langle \xi \rangle B. \end{aligned}$$

\square

5.4 Antichains

We note that Lemma 5.4 indeed recursively reduces the general $<_\xi$ question between worms, to the $<_\xi$ questions between worms in S_ξ .

It is easy to see that $<_\xi$ is not tree-like; for example, we see that both $011 <_1 10111 <_1 1111$ and $011 <_1 11011 <_1 1111$ while 10111 and 11011 are $<_1$ incomparable.

A set of elements $\{A_i\}_{i < \zeta}$ is called an *anti-chain* for $<_\alpha$ if for all $i \neq j$ we have that A_i and A_j are $<_\alpha$ -incomparable. That is, $A_i \not\leq_\alpha A_j$ and $A_j \not\leq_\alpha A_i$. An ordered set $\langle X, \prec \rangle$ is called a *well-quasi order* if X contains no infinite anti-chains for \prec .

For $\alpha > 0$, we have that $<_\alpha$ does not define a well-quasi-ordering on S . For example, all elements $\{\langle \beta \rangle \top \mid \beta < \alpha\}$ are mutually $<_\alpha$ incomparable yielding us an infinite anti-chain. A natural questions to study for the $<_\alpha$ orderings on $S \times S$ concerns the $<_0$ length of anti-chains. So, given a worm A , we can consider sets $X_i = \{B \mid B <_\alpha A\}$ so that all elements in X_i are mutually $<_\alpha$ -incomparable. The question arises, what is $\sup_i \text{ot}(X_i, <_0)$? Or more in general, what is $\sup_i \text{ot}(X_i, <_\beta)$ for $\beta < \alpha$?

More generally, we can ask for the supremum of order-types of the $<_\alpha$ anti-chains that lie in between two $<_\alpha$ comparable elements. For example, the set $\{101, 10101, 10101, \dots\}$ defines an $<_1$ anti-chain of $<_0$ order-type ω between 1 and 11 .

It is important to somehow bound where the anti-chain can reside, if not we get anti-chains of arbitrary length. For example, $\{10\alpha \mid \alpha \in \text{On} \setminus \{0\}\}$ defines an anti-chain w.r.t. the $<_1$ order that is unbounded in the $<_0$ order.

Currently it is not clear how to give a sensible arithmetical interpretation of anti-chains in the Japaridze algebra (if possible at all). We shall briefly outline here that anti-chains do not yield sequences of mutually non-interpretable sentences and refer the reader to for example [13] or [18] for details. Basically this is due to the effect that interpretability and \geq_0 coincide on the class of worms. Let us first fix some notation. By $A \triangleright B$ we denote that $T + A$ interprets $T + B$. That is, there is some structure preserving translation j that maps symbols of T to formulas of arithmetic which transforms every $T + B$ theorem into a $T + A$ theorem:

$$A \triangleright B := \exists j \forall \phi (\Box_{T+B} \phi \rightarrow \Box_{T+A} \phi^j).$$

Lemma 5.5. *For any pair of worms A and B and consistent base theory T w.r.t. which GLP is sound, we have*

$$A \triangleright B \iff A \geq B.$$

Proof. The case that $A = B$ is trivial so we may assume them different. If $T \vdash A \rightarrow \Diamond B$, then we can formalize the Henkin construction so that $\Diamond B$ defines an internal model of $T + B$. This model in its turn defines the translation j , so that we obtain $A \triangleright B$.

Suppose now $A \triangleright B$ but $\neg(A \geq B)$. By linearity of $<_0$ we would get $B > A$, whence $T \vdash B \rightarrow \Diamond A$. Now, using the identity interpretation, we see that

$B \triangleright \diamond A$. By transitivity of interpretability, we get $A \triangleright \diamond A$ which contradicts Fefermans generalization of Gödel's Second Incompleteness Theorem to the effect that no consistent theory can interpret its own consistency. \square

One could easily define a generalized notion of interpretability

$$A \triangleright_n B := \exists j \forall \phi ([n]_{T+B}\phi \rightarrow [n]_{T+A}\phi^j)$$

but it is not clear whether \triangleright_n would coincide with \geq_n on the class of all worms.

6 Omega sequences

In this section we give a full characterization of the sequences $\vec{\Omega}(A)$; that is, we shall determine for given A each of the values $\Omega_\xi(A)$ and classify at what coordinates ξ the $\vec{\Omega}(A)$ sequence changes value.

6.1 Basic properties of omega sequences

Clearly, $\vec{\Omega}(A)$ defines a weakly decreasing sequence of ordinals.

Lemma 6.1. *For $\xi < \zeta$ we have that $\Omega_\xi(A) \geq \Omega_\zeta(A)$.*

Proof. In general we have for $\xi < \zeta$ that $A \rightarrow \langle \zeta \rangle B$ implies $A \rightarrow \langle \xi \rangle B$. Thus, any $<_\zeta$ sequence is automatically also a $<_\xi$ sequence. \square

In particular, since the omega sequences are weakly decreasing on the ordinals, we have that $\{\Omega_\xi(A) \mid \xi \in \mathbf{Ord}\}$ is a finite set for any worm A .

Lemma 6.2. $\Omega_\xi(A) = o_\xi h_\xi(A)$

Proof. Suppose $A_0 <_\xi A_1 <_\xi \dots <_\xi A$, then

$$h_\xi(A_0) <_\xi h_\xi(A_1) <_\xi \dots <_\xi h_\xi(A)$$

by Lemma 5.4 whence $\Omega_\xi(A) \leq o_\xi h_\xi(A)$.

On the other hand, if $B <_\xi h_\xi(A)$, then $h_\xi(A) \rightarrow \langle \xi \rangle B$. But as $A \leftrightarrow h_\xi(A) \wedge r_\xi(A)$ we also have $A \rightarrow \langle \xi \rangle B$. Consequently $o_\xi h_\xi(A) \leq \Omega_\xi(A)$. \square

Corollary 6.3. *For each worm $A \neq \lambda$, there is a maximal ξ so that $\Omega_\xi(A) \neq 0$. In particular we have $\xi = \mathbf{First}(A)$, where $\mathbf{First}(A)$ is the left-most element of A .*

Proof. For $A \in S$, we denote by $\mathbf{First}(A)$ the first element of A , that is, $\mathbf{First}(\lambda) = \lambda$, and $\mathbf{First}(\xi * B) = \xi$. Clearly, $h_{\mathbf{First}(A)}(A) \neq \lambda$ whence by Lemma 6.2,

$$\Omega_{\mathbf{First}(A)}(A) \neq 0.$$

On the other hand, for $\xi > \mathbf{First}(A)$, clearly $h_\xi(A) = \lambda$ whence $\Omega_\xi(A) = 0$. \square

It is good to have reduced $\Omega_\xi(A)$ to $o_\xi(A)$ as in Section 4 we provided a full calculus for it (Lemma 4.9).

Lemma 6.1 and Corollary 6.3 are first simple observations on $\vec{\Omega}(A)$ sequences. In the remainder of this section we shall provide a full characterization of them.

6.2 Successor coordinates

First let us compute $\Omega_{\xi+1}(A)$ in terms of $\Omega_\xi(A)$. Recall that $\ell\alpha$ denotes the unique β such that $\alpha = \alpha' + \omega^\beta$ for $\alpha > 0$. For convenience we define $\ell 0 = 0$. The following lemma will be useful:

Lemma 6.4. *Given an ordinal ξ and a worm A ,*

$$o_{\xi+1}h_{\xi+1}(A) = \ell o_\xi h_\xi(A).$$

Proof. We write $h_\xi(A)$ as $A_0\xi \dots \xi A_n$. Clearly, $h_{\xi+1}(A) = A_0$. We shall now see that $\ell o_\xi h_\xi(A) = o_{\xi+1}(A_0)$.

To this end, we observe that

$$\begin{aligned} o_\xi h_\xi(A) &= o_\xi(A_0\xi \dots \xi A_n) \\ &= o\left((\xi \downarrow A_0)0 \dots 0(\xi \downarrow A_n)\right) \\ &= \omega^{o_1(\xi \downarrow A_n)} + \dots + \omega^{o_1(\xi \downarrow A_0)} \\ &= \omega^{o_{\xi+1}(A_n)} + \dots + \omega^{o_{\xi+1}(A_0)} \end{aligned}$$

Consequently $\ell o_\xi h_\xi(A) = o_{\xi+1}(A_0)$, as desired. \square

Now we are ready to describe the relation between successor coordinates of the $\vec{\Omega}(A)$ sequence.

Theorem 6.5. $\Omega_{\xi+1}(A) = \ell\Omega_\xi(A)$

Proof.

$$\begin{aligned} \Omega_{\xi+1}(A) &= o_{\xi+1}h_{\xi+1}(A) \quad \text{by Lemma 6.4} \\ &= \ell o_\xi h_\xi(A) \\ &= \ell\Omega_\xi(A) \quad \text{by Lemma 5.4.} \end{aligned}$$

\square

Theorem 6.5 tells us what the relation between successor coordinates of $\vec{\Omega}(A)$ is. We may also infer from it when successor coordinates are different; if $\Omega_\xi(A)$ is a fixed point of $\zeta \mapsto \omega^\zeta$ then $\Omega_\xi(A) = \Omega_{\xi+1}(A)$.

6.3 Equal coordinates

Theorem 6.7 below gives us a characterization of when different coordinates attain different or equal values. Before we can state and prove this theorem we first need some notation and background reasoning on CNFs.

For $\alpha \in \text{On}$ we define N_α and the syntactic operation $\text{CNF}(\alpha) := \sum_{i=1}^{N_\alpha} \omega^{\xi_i}$ to be the unique CNF expression of α . Next, we define for an ordinal α the

set of its *Cantor Normal Form Approximations* as the set of partial sums of $\text{CNF}(\alpha)$, that is, if

$$\text{CNF}(\alpha) = \sum_{i=1}^{N_\alpha} \omega^{\xi_i},$$

then

$$\text{CNA}(\alpha) := \left\{ \sum_{i=1}^k \omega^{\xi_i} : 0 \leq k \leq N_\alpha \right\}.$$

We also define the *Cantor Normal Form Projection* of some ordinal ζ on another ordinal ξ as follows:

$$\text{CNP}(\zeta, \xi) := \max\{\xi' \in \text{CNA}(\xi) \mid \xi' \leq \zeta\}.$$

Note that $\text{CNP}(\zeta, \xi)$ is defined for all $\zeta, \xi \in \text{On}$.

For $\alpha, \beta, \gamma \in \text{On}$ we define

$$\alpha \sim_\gamma \beta \iff \text{CNP}(\alpha, \gamma) = \text{CNP}(\beta, \gamma).$$

In words, $\alpha \sim_\gamma \beta$ whenever there is no partial sum of the CNF of γ that falls in between α and β (also the case that both α and β are non-equal partial sums is excluded).

The just-defined notions of $\text{CNA}(\xi)$, $\text{CNP}(\zeta, \xi)$ and $\alpha \sim_\gamma \beta$ are needed to characterize the $\xi \downarrow \zeta$ operation.

Lemma 6.6. *Let ζ, ξ and η be ordinals.*

1. $\forall \zeta \leq \xi \ \zeta \downarrow \xi = \text{CNP}(\zeta, \xi) \downarrow \xi$;
2. $\forall \zeta \leq \xi \exists ! \eta \in \text{CNA}(\xi) \ \zeta \downarrow \xi = \eta \downarrow \xi$;
3. For $\xi, \zeta \leq \eta$, we have $\xi \downarrow \eta = \zeta \downarrow \eta \iff \xi \sim_\eta \zeta$.

Proof. 1. We consider $\zeta \leq \xi$. Now let $\eta = \max\{\eta' \in \text{CNA}(\xi) \mid \eta' \leq \zeta\} = \text{CNP}(\zeta, \xi)$. The claim is that $\zeta \downarrow \xi = \eta \downarrow \xi$. Let

$$\text{CNF}(\xi) = \sum_{i=1}^{N_\xi} \omega^{\xi_i}.$$

As $\eta = \sum_{i=1}^k \omega^{\xi_i}$ for some $k \leq N_\xi$, we see that

$$\eta \downarrow \xi = \sum_{i=k+1}^{N_\xi} \omega^{\xi_i}$$

for $k < N_\xi$ and $\eta \downarrow \xi = 0$ for $k = N_\xi$. We now claim that $\zeta + (\eta \downarrow \xi) = \xi$ so that $\zeta \downarrow \xi = \eta \downarrow \xi$ follows from the fact that

$$\forall \zeta < \xi \exists ! \delta \ \zeta + \delta = \xi.$$

We may assume $\zeta > \eta$ otherwise $\zeta + (\eta \downarrow \xi) = \xi$ is trivial.

Thus,

$$\eta = \sum_{i=1}^k \omega^{\xi_i} < \zeta \leq \sum_{i=1}^{k+1} \omega^{\xi_i}.$$

As by the definition of η we see that $\zeta \leq \sum_{i=1}^{k+1} \omega^{\xi_i}$ cannot be an equality whence

$$\eta = \sum_{i=1}^k \omega^{\xi_i} < \zeta < \sum_{i=1}^{k+1} \omega^{\xi_i}.$$

Thus, $\eta \in \text{CNA}(\zeta)$ and $\zeta + \sum_{i=k+1}^{N_\xi} \omega^{\xi_i} = \xi$, whence

$$\sum_{i=k+1}^{N_\xi} \omega^{\xi_i} = \zeta \downarrow \xi = \sum_{i=1}^k \omega^{\xi_i} = \eta \downarrow \xi.$$

2. Follows from part 1 once we realize that for different η and η' both in $\text{CNA}(\xi)$ we have $\eta \downarrow \xi \neq \eta' \downarrow \xi$.

3. From the proof of part 1 we see that

$$\xi \downarrow \eta = \zeta \downarrow \eta \Leftrightarrow \max\{\eta' \in \text{CNA}(\eta) \mid \eta' \leq \xi\} = \max\{\eta' \in \text{CNA}(\eta) \mid \eta' \leq \zeta\}$$

where the latter is precisely the definition of $\xi \sim_\eta \zeta$. \square

Once we have this lemma to characterize the $\xi \downarrow \zeta$ operation, we are armed to prove a characterization for when two coordinates in $\bar{\Omega}(A)$ are equal.

Theorem 6.7. *The following five conditions are equivalent.*

1. $\Omega_\xi(A) = \Omega_\zeta(A)$
2. $o_\xi h_\xi(A) = o_\zeta h_\zeta(A)$
3. $\xi \downarrow h_\xi(A) = \zeta \downarrow h_\zeta(A)$
4. $h_\xi(A) = h_\zeta(A)$ and $\xi \downarrow h_\xi(A) = \zeta \downarrow h_\zeta(A)$
5. $h_\xi(A) = h_\zeta(A)$ and $\forall \eta \in h_\xi(A)$, $\xi \sim_\eta \zeta$

Proof. (1.) \Leftrightarrow (2.) is just Lemma 6.2.

(2.) \Leftrightarrow (3.): Observe that $o_\xi(h_\xi(A)) = o(\xi \downarrow h_\xi(A))$ and $o_\zeta(h_\zeta(A)) = o(\zeta \downarrow h_\zeta(A))$. As o defines an isomorphism between S and On , we obtain

$$o_\xi h_\xi(A) = o_\zeta h_\zeta(A) \Leftrightarrow \xi \downarrow h_\xi(A) = \zeta \downarrow h_\zeta(A).$$

(3.) \Leftrightarrow (4.): Suppose $\xi \downarrow h_\xi(A) = \zeta \downarrow h_\zeta(A)$. Then, it follows that the two have equal length; further, they have length equal to that of $h_\xi(A), h_\zeta(A)$, respectively. But two initial segments of A of equal length must be equal, that is, $h_\xi(A) = h_\zeta(A)$.

(4.) \Leftrightarrow (5.):

$$\begin{aligned} h_\xi(A) = h_\zeta(A) & \quad \& \quad \xi \downarrow h_\xi(A) = \zeta \downarrow h_\zeta(A) & \quad \Leftrightarrow \\ h_\xi(A) = h_\zeta(A) & \quad \& \quad \forall \eta \in h_\xi(A) \xi \downarrow \eta = \zeta \downarrow \eta & \quad \Leftrightarrow \quad \text{by Lemma 6.6.3} \\ h_\xi(A) = h_\zeta(A) & \quad \& \quad \forall \eta \in h_\xi(A) \xi \sim_\eta \zeta \end{aligned}$$

□

6.4 Limit coordinates

The results so far have already provided us with quite some insight about what the sequences $\vec{\Omega}(A)$ look like. By Lemma 6.1 we know that the set of values that occur in $\vec{\Omega}(A)$ is finite. Moreover, by Theorem 6.5 we know exactly the values at successor coordinates. In particular, we know that if the value of $\vec{\Omega}(A)$ at ξ is the same as at the successor coordinate, then it remains the same for all further successors.

The question remains what happens at limit ordinals coordinates. In this subsection we shall determine at what limit ordinals a new value can be attained and how the new value relates to previous values. Let us start out the analysis by formulating a negative version of Theorem 6.7.

Lemma 6.8. *For $\zeta < \xi$ we have that*

$$\begin{aligned} \Omega_\zeta(A) > \Omega_\xi(A) & \Leftrightarrow \\ (\exists \eta \in h_\zeta(A) \zeta \leq \eta < \xi) \vee (\exists \eta \in h_\zeta(A) \text{CNP}(\zeta, \eta) < \text{CNP}(\xi, \eta)). \end{aligned}$$

Proof. By contraposing equivalence (1.) \Leftrightarrow (5.) of Theorem 6.7 we get

$$\Omega_\zeta(A) \neq \Omega_\xi(A) \Leftrightarrow h_\xi(A) \neq h_\zeta(A) \vee \exists \eta \in h_\zeta(A) \xi \not\sim_\eta \zeta.$$

But, as $\zeta < \xi$ we see

$$h_\xi(A) \neq h_\zeta(A) \Leftrightarrow \exists \eta \in h_\zeta(A) \zeta \leq \eta < \xi.$$

Likewise,

$$\exists \eta \in h_\zeta(A) \xi \not\sim_\eta \zeta \Leftrightarrow \exists \eta \in h_\zeta(A) \text{CNP}(\zeta, \eta) \neq \text{CNP}(\xi, \eta).$$

As $\zeta < \xi$ we have

$$\text{CNP}(\zeta, \eta) \neq \text{CNP}(\xi, \eta) \Leftrightarrow \text{CNP}(\zeta, \eta) < \text{CNP}(\xi, \eta).$$

□

The first question to ask is at which limit coordinates the sequence $\vec{\Omega}(A)$ can change. Let us first write precisely what it means for the sequence $\vec{\Omega}(A)$ to change at some coordinate ζ . We express this by the expression

$$\text{Change}(\zeta, A) := \exists \xi < \zeta (\Omega_\xi(A) > \Omega_\zeta(A) \ \& \ \forall \eta (\xi \leq \eta < \zeta \Rightarrow \Omega_\xi(A) = \Omega_\zeta(A))).$$

The next lemma gives an alternative characterization of $\text{Change}(\zeta, A)$.

Lemma 6.9. $\text{Change}(\zeta, A) \Leftrightarrow \forall \xi < \zeta \ \Omega_\xi(A) > \Omega_\zeta(A)$

Proof. For $\zeta \in \text{Succ}$ this is clear. If $\zeta \in \text{Lim}$, then $\{\Omega_\xi(A) \mid \xi < \zeta\}$ is a finite set as all the $\Omega_\xi(A) \in \text{On}$ and these are weakly decreasing. Thus, at some point below ζ the sequence must stabilize. \square

We can now characterize at what limit ordinals the sequence $\vec{\Omega}(A)$ can change.

Theorem 6.10. For $\zeta \in \text{Lim}$: $\text{Change}(\zeta, A) \Leftrightarrow \exists \xi \in h_\zeta(A) \ \zeta \in \text{CNA}(\xi)$

Proof. For $\zeta \in \text{Lim}$ we see that, by Lemma 6.9, $\text{Change}(\zeta, A)$ is equivalent to the claim that, given $\xi < \zeta$, $\Omega_\xi(A) > \Omega_\zeta(A)$.

By Lemma 6.8, the latter is in turn equivalent to

$$\forall \xi < \zeta (\exists \eta \in h_\xi(A) \ \xi \leq \eta < \zeta \ \vee \ \exists \eta \in h_\xi(A) \ \text{CNP}(\xi, \eta) < \text{CNP}(\zeta, \eta)),$$

or equivalently,

$$\forall \xi (\xi_0 < \xi < \zeta \rightarrow \exists \eta \in h_\zeta(A) \ \text{CNP}(\xi, \eta) < \text{CNP}(\zeta, \eta)),$$

where $\xi_0 := \max\{\xi' \in A \mid \xi' < \zeta\}$. Note that for these ξ , indeed, we have $h_\xi(A) = h_\zeta(A)$. We now claim that the latter is equivalent to $\exists \eta \in h_\zeta(A) \ \zeta \in \text{CNA}(\eta)$. Clearly, if $\zeta \in \text{CNA}(\eta)$ for some $\eta \in h_\zeta(A)$, then $\xi \downarrow \eta < \zeta \downarrow \eta$ for each $\xi < \zeta$.

For the converse direction, suppose $\zeta \notin \text{CNA}(\eta)$ for all $\eta \in h_\zeta(A)$. Then, for all ξ' with

$$\max \bigcup_{\eta \in h_\zeta(A)} \text{CNA}(\eta) < \xi' < \zeta$$

we have $\xi' \sim_\eta \zeta$ for all $\eta \in h_\zeta(A)$, whence by Theorem 6.7 $\Omega_{\xi'}(A) = \Omega_\zeta(A)$. \square

Now that we have fully determined at which limit coordinates a change can occur the only thing left to establish is the size of the change. In other words, if $\text{Change}(\zeta, A)$ for some $\zeta \in \text{Lim}$, how does $\Omega_\zeta(A)$ relate to $\Omega_\xi(A)$ for $\xi < \zeta$?

Here, our functions e^ξ come back into play:

Theorem 6.11. Let $\zeta \in \text{Lim}$, and let $\xi < \zeta$ be such that, whenever $\xi' \in [\xi, \zeta)$, it follows that $\Omega_{\xi'}(A) = \Omega_\xi(A)$. Then, for $\theta \in [\xi, \zeta)$ large enough we have that

$$\Omega_\theta(A) = e^{-\theta+\zeta} \Omega_\zeta(A) = e_{\ell\zeta} \Omega_\zeta(A).$$

Proof. As the values of $\Omega_{\xi'}(A)$ do not change for $\xi \leq \xi' < \zeta$ we know in particular by Theorem 6.7 that $h_{\xi}(A) = h_{\xi'}(A)$ whence also

$$h_{\xi'}(A) = h_{\zeta}(A) \quad \text{for each } \xi' \in [\xi, \zeta]. \quad (3)$$

As $\zeta = \zeta' + \omega^{\ell\zeta}$ we have that $-\xi + \zeta \geq \omega^{\ell\zeta}$. So certainly $-\theta + \zeta = \omega^{\ell\zeta}$ for $\theta \in [\xi, \zeta)$ large enough. Let $\delta = -\theta + \zeta = \theta \downarrow \zeta = \omega^{\ell\zeta}$.

Then,

$$\begin{aligned} \Omega_{\theta}(A) &= \text{Lemma 6.2} \\ o_{\theta}h_{\theta}(A) &= \text{By (3)} \\ o_{\theta}h_{\zeta}(A) &= \text{Lemma 4.9.4} \\ o(\theta \downarrow h_{\zeta}(A)) &= \text{Lemma 4.6.6} \\ o(\delta \uparrow (\zeta \downarrow h_{\zeta}(A))) &= \text{Lemma 4.9.3} \\ e^{\delta}o(\zeta \downarrow h_{\zeta}(A)) &= \text{Lemma 4.9.4} \\ e^{\delta}o_{\zeta}h_{\zeta}(A) &= \text{Lemma 6.2} \\ e_{\ell\zeta}\Omega_{\zeta}(A). & \end{aligned}$$

□

Note that this theorem establishes the size of limit coordinates both in case a change does occur and in case no change occurs. The latter case can only be so when $\Omega_{\zeta}(A)$ is a fixed point of $e_{\ell\zeta}$.

7 From local to global

The previous section has established exactly where changes occur in the $\vec{\Omega}(A)$ sequences. Moreover, it established the size of each change in the sequence. We have distinguished two cases: successor coordinates and limit coordinates.

In Theorem 6.11 we have seen that the value of a limit coordinate fully determines its ‘direct predecessor’ and vice versa. Recall that the value of a successor coordinate is fully determined by the value of its predecessor but not vice versa. Thus, the values of the early coordinates fully determines what comes after it but not so in the other direction. In Section 4 we provided a calculus to compute $o(A)$ for given A . Thus, the results in this section provides sufficient results to fully calculate $\vec{\Omega}(A)$.

However, the algorithm implicit in the current results are of a nature that all computations are performed globally: If we wish to compute $\Omega_{\zeta}(A)$, we need to compute the values of all its predecessors. Thus, first we compute $\Omega_0(A) = o_0(A)$, next we determine at what coordinates the sequence $\vec{\Omega}(A)$ changes up to ζ . In the end we compute all the successive values of the coordinates where $\vec{\Omega}(A)$ changes to finally obtain $\Omega_{\zeta}(A)$.

We shall now see that each change in $\vec{\Omega}(A)$ is of similar nature so that successively computing the changes corresponds to a certain transfinite iteration. Recall that $\Omega_{\xi+1}(A) = \ell\Omega_{\xi}(A)$ by Theorem 6.5. We can see ℓ as a natural left

inverse of $e^1 = e^{-\xi+(\xi+1)}$ so that

$$\begin{aligned} e^1 \Omega_{\xi+1}(A) &= \Omega_{\xi}(A) \\ &\Rightarrow \\ \ell^1 e^1 \Omega_{\xi+1}(A) &= \ell^1 \Omega_{\xi}(A) \\ &\Rightarrow \\ \Omega_{\xi+1}(A) &= \ell \Omega_{\xi}(A). \end{aligned}$$

If, more generally, for every ϑ we find an analogous left inverse ℓ^ϑ for e^ϑ , then we may similarly obtain

$$\begin{aligned} e^{-\xi+\zeta} \Omega_{\zeta}(A) &= \Omega_{\xi}(A) \\ &\Rightarrow \\ \ell^{-\xi+\zeta} e^{-\xi+\zeta} \Omega_{\zeta}(A) &= \ell^{-\xi+\zeta} \Omega_{\xi}(A) \\ &\Rightarrow \\ \Omega_{\xi+1}(A) &= \ell^{-\xi+\zeta} \Omega_{\xi}(A) \end{aligned}$$

when ζ, ξ and A are as in Theorem 6.11.

In [7] the authors systematically study natural left-inverses of hyperations and call them *cohyperations*. Once this is in place we can give a global calculus for our sequences, that is, a calculus that computes $\Omega_{\zeta}(A)$ in ‘one step’ from $\Omega_0(A)$ or from any other previous coordinate.

8 Hyperations and Cohyperations

In this section we shall briefly state the main definitions and results from [7] which are relevant for the current paper. With these at hand we can give a useful characterization of cohyperating the end-exponent function ℓ .

8.1 Hyperations

Hyperation is a form of transfinite iteration of normal functions. It is based on the additivity of finite iterations, that is $f^{m+n} = f^m \circ f^n$ generalizing this to the transfinite setting.

Definition 8.1 (Weak hyperation). *A weak hyperation of a normal function f is a family of normal functions $\langle g^\xi \rangle_{\xi \in \mathcal{O}_n}$ such that*

1. $g^0 \xi = \xi$ for all ξ ,
2. $g^1 = f$,
3. $g^{\xi+\zeta} = g^\xi g^\zeta$.

Par abuse de langage we will often write just g^ξ instead of $\langle g^\xi \rangle_{\xi \in \mathcal{O}_n}$. Weak hyperations are not unique. However, if we impose a minimality condition, we can prove that there is a unique minimal hyperation.

Definition 8.2 (Hyperation). *A weak hyperation g^ξ of f is minimal if it has the property that, whenever h^ξ is a weak hyperation of f and ξ, ζ are ordinals, then $g^\xi \zeta \leq h^\xi \zeta$.*

If f has a (unique) minimal weak hyperation, we call it the hyperation of f and denote it f^ξ .

Hyperations allow for an explicit recursive definition very much in the style of Theorem 4.8. Moreover, there turns out to be a close connection between hyperations and Veblen progressions as shown by the following two theorems.

Theorem 8.3. *Let f be a normal function and let f_α be the Veblen progression based on it. Given an ordinal α , we have that $f^{\omega^\alpha} = f_\alpha$.*

Theorem 8.4. *Let g^ξ be a weak hyperation of a normal function f . If we moreover have that $g^{\omega^\alpha} = f_\alpha$ for each α then $g^\xi = f^\xi$.*

We will call the functions e^α hyperexponentials. They can be used to define weak normal forms. For example, given an ordinal ξ , we say an expression

$$\xi = \sum_{i < I} e^{\alpha_i} \beta_i + n$$

is a *Weak Hyperexponential Normal Form* if $I, n < \omega$, and for each $i + 1 < I$, both $e^{\alpha_i} \beta_i \geq e^{\alpha_{i+1}} \beta_{i+1}$ and $\beta_i < e^{\alpha_i} \beta_i$. Note that Weak Hyperexponential Normal Forms are typically not unique. For example $\omega^\omega = e^2 1 = e^1 \omega$. We do, however, have uniqueness if every α_i is of the form ω^δ .

Lemma 8.5. *Every ordinal $\xi > 0$ has a weak hyperexponential normal form.*

If we further require that every exponent be of the form ω^δ , then the WHNF obtained is unique.

Proof. Write ξ in Veblen Normal Form and replace $\varphi_\alpha(\beta)$ by $e^{\omega^\alpha}(1 + \beta)$ for $\alpha > 0$, $\varphi_0(\beta)$ by $e^1(\beta)$ for $\beta > 0$. The occurrences of $\varphi_0(0)$ can be captured in the term $+n$ in the end of a WHNF.

If all exponents are of the form ω^δ , we may invert the process to obtain a VNF from a given WHNF; the uniqueness of the latter follows from the uniqueness of the former. \square

8.2 Cohyperations

Hyperations are injective and hence invertible on the left; however, the inverse of a hyperation is typically not a hyperation, but a different form of transfinite iteration we call *cohyperation*. Instead of iterating normal functions we shall consider *initial functions*. We will say a function g is *initial* if, whenever I is an initial segment (i.e., of the form $[0, \beta)$ for some β), then $f(I)$ is an initial segment. It is easy to see that $f\xi \leq \xi$ for initial functions f .

Definition 8.6 (Cohyperation). *A weak cohyperation of an initial function f is a family of initial functions $\langle g^\xi \rangle_{\xi \in \text{On}}$ such that*

1. $g^0\xi = \xi$ for all ξ ,
2. $g^1 = f$,
3. $g^{\xi+\zeta} = g^\zeta g^\xi$.

If g is maximal in the sense that $g^\xi\zeta \geq h^\xi\zeta$ for every weak cohyperation h of f and all ordinals ξ, ζ , we say g is the cohyperation of f and write $f^\xi = g^\xi$.

Both hyperations and cohyperations are denoted using the superscript; however, this does not lead to a clash in notation as the only function that is both normal and initial is the identity.

There is a general recursive scheme to compute actual cohyperations in the spirit of Definition 4.8.

Lemma 8.7. *Every initial function f has a unique cohyperation, given by*

1. $f^0\alpha = \alpha$,
2. $f^1 = f$,
3. $f^{\omega^\rho+\xi} = f^\xi f^{\omega^\rho}$ provided $\xi < \omega^\rho + \xi$,
4. $f^{\omega^\rho}\xi = f^{\omega^\rho} f^\eta\xi$, if $f^\eta\xi < \xi$ and $\eta < \omega^\rho$,
5. $f^{\omega^\rho}\xi = \sup_{\zeta < \xi} (f^{\omega^\rho}\zeta + 1)$, if $f^\eta\xi = \xi$ for all $\eta < \omega^\rho$, with $\rho > 0$.

At first glance it is not even clear that f^ξ is well defined in that it is single valued. In Item 4., there might be various η 's below ω^ρ so that $f^\eta\xi < \xi$. In [7] it is shown that it does not matter which η one takes.

Let f be a normal function. Then, g is a *left adjoint* for f if, for all ordinals α, β ,

1. if $\alpha = f(\beta)$, then $g(\alpha) = \beta$ and
2. if $\alpha < f(\beta)$, then $g(\alpha) < \beta$.

Left-adjoints are natural left-inverses and cohyperating them yields left-adjoints to the corresponding hyperations in a uniform way:

Theorem 8.8. *Given a normal function f with left adjoint g and ordinals $\xi < \zeta$ and α , $g^\xi f^\zeta = f^{-\xi+\zeta}$ and $g^\zeta f^\xi = g^{-\xi+\zeta}$.*

Theorem 8.9. *The function ℓ is a left adjoint to e , and thus ℓ^ξ is left adjoint to e^ξ for all ξ .*

For the cohyperation of ℓ we give the following easy recursive scheme.

Theorem 8.10. *For ordinals ξ, ζ , the value of $\ell^\xi\zeta$ is given by the following recursion:*

1. $\ell^0\alpha = \alpha$,
2. $\ell^\xi n = 0$ for $n \in \omega$ and $\xi > 0$,

3. $\ell^\xi(\alpha + \omega^\beta) = \ell^\xi \omega^\beta$ if $\xi > 0$,
4. $\ell^{\omega^\rho + \xi} = \ell^\xi \ell^{\omega^\rho}$ provided $\xi < \omega^\rho + \xi$,
5. $\ell^{\omega^\rho} e^{\omega^\beta} \xi = \begin{cases} e^{\omega^\beta} \xi & \text{if } \omega^\rho < \omega^\beta, \\ \xi & \text{if } \omega^\rho = \omega^\beta, \\ \ell^{\omega^\rho} \xi & \text{in case } \omega^\rho > \omega^\beta. \end{cases}$

Proof. We shall first see that the recursive scheme of the unique cohyperation of ℓ as given in Lemma 8.7 satisfies the recursion of the current theorem. Next, we shall see that the recursion of this theorem has a unique solution. The latter is necessary as we note that it is not fully determined how the last item of the recursion is to be applied, as an ordinal ζ might be representable as $e^{\omega^\beta} \xi$ in various ways using different β and ξ .

That $\ell^0 \alpha = \alpha$ follows directly from Lemma 8.7. Any $\xi > 0$ can be written as $1 + \xi'$ so that

$$\ell^\xi(\alpha + \omega^\beta) = \ell^{1+\xi'}(\alpha + \omega^\beta) = \ell^{\xi'} \ell(\alpha + \omega^\beta) = \ell^{\xi'} \ell \omega^\beta = \ell^{\xi'} \omega^\beta = \ell^{1+\xi'} \omega^\beta = \ell^\xi \omega^\beta.$$

From this, it directly follows that $\ell^\xi n = 0$ for any $\xi > 0$ and $n \in \omega$. Item 4 of the recursion holds trivially. Item 5 follows directly from Theorem 8.9 and Theorem 8.8.

We shall now show unicity. It is clear that we only need to focus on Item 5. Thus, we consider $\ell^{\omega^\rho} e^{\omega^\beta} \xi$. In [14] it is shown that there is a maximal α such that $e^{\omega^\beta} \xi = e^\alpha \zeta$ for some ζ . We shall prove that $\ell^{\omega^\rho} e^{\omega^\beta} \xi = \ell^{\omega^\rho} e^\alpha \zeta$.

Let $\omega^{\alpha_1} + \dots + \omega^{\alpha_n} =_{\text{CNF}} \alpha$ for this particular α . By maximality of α , we see that $\beta \leq \alpha_1$. In case $\beta < \alpha_1$ we see by Theorem 8.3 that $e^{\omega^\alpha} \zeta$ is a fixpoint of e^{ω^β} so that

$$e^{\omega^\beta} \xi = e^\alpha \zeta = e^{\omega^\beta} e^{\omega^\alpha} \zeta = e^{\omega^\alpha} \zeta,$$

whence also $\ell^{\omega^\rho} e^{\omega^\beta} \xi = \ell^{\omega^\rho} e^\alpha \zeta$.

In case $\beta = \alpha_1$, we see also have that $\ell^{\omega^\rho} e^{\omega^\beta} \xi = \ell^{\omega^\rho} e^\alpha \zeta$ as $e^{\omega^{\alpha_1}}$ is injective. \square

We will refer to the functions ℓ^ξ as *hyperlogarithms*.

8.3 Exact sequences

A nice feature of cohyperations is that, in a sense, they need only be defined locally. To make this precise, we introduce the notion of an *exact sequence*.

Definition 8.11. *Let g^ξ be a cohyperation, and $f : \Lambda \rightarrow \Theta$ be an ordinal function.*

Then, we say f is g -exact if, given ordinals ξ, ζ with $\xi + \zeta < \Lambda$, $f(\xi + \zeta) = g^\zeta f(\xi)$.

A g -exact function f describes the values of $g^\xi f(0)$. However, for f to be g -exact, we need only check a fairly weak condition:

Lemma 8.12. *The following are equivalent:*

1. f is g -exact
2. for every ordinal ξ , $f(\xi) = g^\xi f(0)$
3. for every ordinal $\zeta > 0$ there is $\xi < \zeta$ such that $f(\zeta) = g^{-\xi+\zeta} f(\xi)$.

9 A global characterization

In this section we shall unify the results obtained so far by describing the sequences $\vec{\Omega}(A)$ using hyperexponentials and $-$ -logarithms.

Theorem 9.1. *Let A be a worm.*

Then, $\vec{\Omega}(A)$ is the unique ℓ -exact sequence with $\Omega_0(A) = o(A)$.

Proof. In view of Lemma 8.12, it suffices to show that, given any ordinal ζ , there is $\xi < \zeta$ such that $\Omega_\zeta(A) = \ell^{-\xi+\zeta} \Omega_\xi(A)$.

If ζ is a successor ordinal, write $\zeta = \xi + 1$. Then, by Theorem 6.5, we have that $\Omega_\zeta(A) = \ell \Omega_\xi(A)$.

Meanwhile, if ζ is a limit ordinal, we know from Lemma 6.11 that, for $\xi < \zeta$ large enough,

$$\Omega_\xi(A) = e^{-\xi+\zeta} \Omega_\zeta(A).$$

Applying $\ell^{-\xi+\zeta}$ on both sides and using Theorem 8.8, we see that

$$\ell^{-\xi+\zeta} \Omega_\xi(A) = \Omega_\zeta(A).$$

Thus we can use Lemma 8.12 to see that $\vec{\Omega}(A)$ is ℓ -exact, so that, for all ξ ,

$$\Omega_\xi(A) = \ell^\xi \Omega_0(A) = \ell^\xi o_0(A),$$

as claimed. □

Notice by Theorems 9.1 and 8.10 that the computations in omega sequences are rather easy if we have written the values in Weak Hyperexponential Normal Form (see Lemma 8.5) and are determined by the last term. If, for example, $\Omega_\xi(A) = \alpha + e^{\omega^\xi}(\beta)$, then the next value where the $\vec{\Omega}(A)$ sequence changes will be in $\xi + \omega^\xi$ jumping to the new value $\Omega_{\xi+\omega^\xi}(A) = \beta$.

Further, hyperexponentials give us *lower bounds* on ℓ -exact sequences. The value of $\Omega_\xi(A)$ fully determines the values of $\Omega_\zeta(A)$ for $\zeta > \xi$ but not vice versa. However for $\zeta > \xi$ we do have a lower-bound on $\Omega_\zeta(A)$:

Theorem 9.2. *Given a worm A and ordinals ξ, ζ , $\Omega_\xi(A) \geq e^\zeta \Omega_{\xi+\zeta}(A)$.*

Proof. Towards a contradiction, assume that there is a worm A and ordinals $\xi < \zeta$ such that $\Omega_\xi(A) < e^{-\xi+\zeta} \Omega_\zeta(A)$. Then, by Theorem 8.9, $\ell^{-\xi+\zeta} \Omega_\xi(A) < \Omega_\zeta(A)$.

But this is impossible by Theorem 9.1, given that $\ell^{-\xi+\zeta} \Omega_\xi(A) = \Omega_\zeta(A)$. □

10 Turing progressions revisited

In this section we shall interpret our omega sequences in GLP_ω in terms of Turing progressions. Before doing so, we first need to introduce a slightly generalized notion of Turing progressions where we transfinitely iterate i -consistency rather than normal consistency:

$$\begin{aligned} T_0^i &:= T; \\ T_{\alpha+1}^i &:= T_\alpha^i \cup \{\langle i \rangle_{T_\alpha^i} \top\}; \\ T_\lambda &:= \bigcup_{\alpha < \lambda} T_\alpha \quad \text{for limit } \lambda. \end{aligned}$$

In this section we shall always assume that GLP_ω is sound w.r.t. the base theory T . Generalized Turing progressions are not sensitive to adding “small” elements to the base theory as is expressed by the following lemma.

Lemma 10.1. *For T an elementary presented theory and for any GLP_ω worm A , if $m < n$, then*

$$(T + mA)_\alpha^n \equiv (T)_\alpha^n + mA \quad \text{for any } \alpha < \epsilon_0.$$

Proof. By transfinite induction on α . The only interesting case is at successor ordinals.

$$\begin{aligned} (T + mA)_{\alpha+1}^n &\equiv_{\text{def}} (T + mA)_\alpha^n + \langle n \rangle_{(T+mA)_\alpha^n} \top \\ &\equiv_{\text{IH}} T_\alpha^n + mA + \langle n \rangle_{T_\alpha^n + mA} \top \\ &\equiv T_\alpha^n + mA + \langle n \rangle_{T_\alpha^n} (\top \wedge mA) \quad \text{by Lemma 1.3} \\ &\equiv T_\alpha^n + mA + \langle n \rangle_{T_\alpha^n} (\top) \\ &\equiv T_{\alpha+1}^n + mA \end{aligned}$$

□

We shall need a generalization of Proposition 2.2 which can be found in [3]. In this section, $U \equiv_n V$ will denote that the theories U and V prove exactly the same Π_{n+1} sentences.

Lemma 10.2. *Let T be some elementary presented theory containing EA^+ whose axioms have logical complexity at most Π_{n+1} and let A be some worm in S_n . We have that*

$$T + A \equiv_n T_{o_n(A)}^n.$$

In general we do of course not have that if $U \equiv_n V$, then $U + \psi \equiv_n V + \psi$ for theories U and V and formulas ψ . However, in the case of Turing progression we can add “small” additions on both sides and preserve conservativity.

Lemma 10.3. *Let T be some elementary presented theory containing EA^+ whose axioms have logical complexity at most Π_{n+1} and let A be some worm in S_n . Moreover, let B be any worm and $m < n$. We have that*

$$T + A + mB \equiv_n T_{o_n(A)}^n + mB.$$

Proof. As $m < n$ we have that $mB \in \Pi_n$. Whence, we can apply Lemma 10.2 to the theory $T + mB$ and obtain

$$T + mB + A \equiv_n (T + mB)_{o_n(A)}^n$$

However, by Lemma 10.1 we see that

$$(T + mB)_{o_n(A)}^n \equiv T_{o_n(A)}^n + mB, \quad \text{whence } T + mB + A \equiv_n T_{o_n(A)}^n + mB.$$

□

From this lemma we obtain the following simple but very useful corollary.

Corollary 10.4. *Let T be some elementary presented theory containing EA^+ whose axioms have logical complexity at most Π_{n+1} . Moreover, let A be any worm. We have that*

$$T + A \equiv_n T_{\Omega_n(A)}^n + r_n(A).$$

Proof. We know that $\text{GLP} \vdash A \leftrightarrow h_n(A) \wedge r_n(A)$. As by assumption GLP_ω is sound w.r.t. T we see that

$$T + A \equiv T + h_n(A) + r_n(A).$$

The worm $r_n(A)$ is either empty or of the form mA for some $m < n$. Clearly, $h_n(A) \in S_n$. Thus, we can apply Lemma 10.3 and obtain

$$T + h_n(A) + r_n(A) \equiv_n T_{o_n(h_n(A))}^n + r_n(A).$$

However, by Lemma 6.2 we know that $o_n(h_n(A)) = \Omega_n(A)$ and we are done. □

From Lemma 10.2 we see that we can capture the Π_1^0 consequences of the $o(A)$ -th Turing Progression of T by the simply axiomatized theory $T + A$. Thus, $T + A$ proves the same Π_1^0 formulas as $T_{o(A)}^0$. However, $T + A$ will in general prove many new formulas of higher complexity. We can characterize those consequences of $T + A$ also in terms of Turing progressions and the way to do so is simply given by our Ω -sequences.

Theorem 10.5. *Let T be some Π_1^0 axiomatizable elementary representable theory containing EA^+ . Let A be any GLP_ω worm. We have that*

$$T + A \equiv \bigcup_{i < \omega} T_{\Omega_i(A)}^i.$$

Proof. We prove by induction on n that

$$T + A \equiv_n \bigcup_{i=0}^n T_{\Omega_i(A)}^i.$$

This is clearly sufficient as for any GLP_ω worm A there are only finitely non-zero entries in $\vec{\Omega}(A)$. The base case follows directly from Lemma 10.2 since $\Omega_0(A) = o_0(A)$.

For the inductive case we reason as follows. By Corollary 10.4 we know that

$$T + A \equiv_{n+1} T_{\Omega_{n+1}(A)}^{n+1} + r_{n+1}(A). \quad (4)$$

In particular, as $T_{\Omega_{n+1}(A)}^{n+1} + r_{n+1}(A) \subseteq \Pi_{n+2}$ we see that actually, $T + A$ is a Π_{n+2} -conservative extension of $T_{\Omega_{n+1}(A)}^{n+1} + r_{n+1}(A)$, and

$$T + A \vdash T_{\Omega_{n+1}(A)}^{n+1} + r_{n+1}(A).$$

The induction hypothesis tells us that

$$T + A \equiv_n \bigcup_{i=0}^n T_{\Omega_i(A)}^i. \quad (5)$$

Again, since $\bigcup_{i=0}^n T_{\Omega_i(A)}^i \subseteq \Pi_{n+1}$ we obtain that

$$T + A \vdash \bigcup_{i=0}^n T_{\Omega_i(A)}^i.$$

Thus, $T + A \vdash \bigcup_{i=0}^{n+1} T_{\Omega_i(A)}^i$ and in particular, if $\bigcup_{i=0}^{n+1} T_{\Omega_i(A)}^i \vdash \pi$ then $T + A \vdash \pi$ for $\pi \in \Pi_{n+2}$.

Conversely, assume that $T + A \vdash \pi$ for some Π_{n+2} sentence π . By (4) we see that $T_{\Omega_{n+1}(A)}^{n+1} + r_{n+1}(A) \vdash \pi$. However, $r_{n+1}(A) \in \Pi_{n+1}$ and $T + A \vdash r_{n+1}(A)$ so, by (5) we see that $\bigcup_{i=0}^n T_{\Omega_i(A)}^i \vdash r_{n+1}(A)$. Thus

$$\begin{array}{l} \bigcup_{i=0}^{n+1} T_{\Omega_i(A)}^i \vdash T_{\Omega_{n+1}(A)}^{n+1} + r_{n+1}(A) \\ \vdash \pi. \end{array}$$

as was required. □

In order to obtain a generalization of Theorem 10.5 for worms A in GLP_Λ for $\Lambda > \omega$ one first would need suitable (hyper)arithmetical interpretations for which GLP_Λ is sound and complete. One such interpretation is given in [10].

References

- [1] L. D. Beklemishev, D. Fernández-Duque, and J. J. Joosten. On provability logics with linearly ordered modalities. forthcoming, 2012.
- [2] L.D. Beklemishev. Provability algebras and proof-theoretic ordinals, I. *Annals of Pure and Applied Logic*, 128:103–124, 2004.
- [3] L.D. Beklemishev. Reflection principles and provability algebras in formal arithmetic. *Uspekhi Matematicheskikh Nauk*, 60(2):3–78, 2005. In Russian. English translation in: *Russian Mathematical Surveys*, 60(2): 197–268, 2005.

- [4] L.D. Beklemishev. Veblen hierarchy in the context of provability algebras. In P. Hájek, L. Valdés-Villanueva, and D. Westerstahl, editors, *Logic, Methodology and Philosophy of Science, Proceedings of the Twelfth International Congress*. Kings College Publications, 2005.
- [5] L.D. Beklemishev. The worm principle. In Z. Chatzidakis, P. Koepke, and W. Pohlers, editors, *Logic Colloquium 2002, Lecture Notes in Logic 27*. ASL Publications, 2006.
- [6] D. Fernández-Duque and J. J. Joosten. Models of transfinite provability logics. *Journal of Symbolic Logic*, 2012. Accepted for publication.
- [7] Fernández-Duque, D. and Joosten, J. J. Hyperations, Veblen progressions and transfinite iteration of ordinal functions. Submitted, May 2012.
- [8] Fernández-Duque, D. and Joosten, J. J. Turing progressions and their well-orders. In *How the world computes*, Lecture Notes in Computer Science, pages 212–221. Springer, 2012.
- [9] Fernández-Duque, D. and Joosten, J. J. Well-orders in the transfinite Japaridze algebra. submitted, 2012.
- [10] Fernández-Duque, D. and Joosten, J. J. Hyperarithmetical interpretations of transfinite provability logic. forthcoming, 2013.
- [11] K. N. Ignatiev. On strong provability predicates and the associated modal logics. *The Journal of Symbolic Logic*, 58:249–290, 1993.
- [12] G. Japaridze. The polymodal provability logic. In *Intensional logics and logical structure of theories: material from the Fourth Soviet-Finnish Symposium on Logic*. Metsniereba, Telavi, 1988. In Russian.
- [13] J. J. Joosten. *Intepretability Formalized*. PhD thesis, Utrecht University, 2004.
- [14] J. J. Joosten. Veblen and the worm. forthcoming, 2013.
- [15] R. M. Solovay. Provability interpretations of modal logic. *Israel Journal of Mathematics*, 28:33–71, 1976.
- [16] A. Turing. Systems of logics based on ordinals. *Proceedings of the London Mathematical Society*, 45:161–228, 1939.
- [17] O. Veblen. Continuous increasing functions of finite and transfinite ordinals. *Transactions of the American Mathematical Society*, 9:280–292, 1908.
- [18] A. Visser. An overview of interpretability logic. In M. Kracht, M. de Rijke, and H. Wansing, editors, *Advances in modal logic '96*, pages 307–359. CSLI Publications, Stanford, CA, 1997.