Semantics for sub-intuitionistic logics

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Abstract

This paper exposes semantics for various sub-intuitionistic logics. The semantics transparently reflect how assumptions on the epistemic and cognitive abilities of the creative subject influences the underlying logic. One of these semantics is used to obtain a lower bound on the length of proofs of certain intuitionistic tautologies.

1 Introduction

This paper is an exposition and exploration of various semantics for different logics. Most logics considered here are subsystems of intuitionistic logic. The idea is that a semantics is developed such that all but one or two of the axioms schemes or rules of intuitionistic logic hold on it.

The motivation for such a project originally came from proof complexity. In particular, the semantics that are found and exposed can be used to pursue lower bounds on the length of proofs in propositional intuitionistic logic. And, as a matter of fact, we employ one particular semantics to prove a linear lower bound for intuitionistic logic. Surely, a linear lower bound is not impressive at all, but it nicely demonstrates how semantics for sub-intuitionistic logics can be used to obtain lower bounds. This method was first applied in a recent paper ([2]) where an exponential lower bound for a large family of modal logics is obtained.

This paper is rather self contained. Whatever is not explicitly mentioned here and concerns intuitionistic logic, can be found in [7] or in [8]. We do not only have technical applications in mind in this paper. The semantics also transparently reflect how assumptions on the epistemic/cognitive abilities of the creative subject influences the underlying logic.

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2 A semantic approach to lower bounds

The motivation for proof complexity comes largely from computational complexity and telling computational complexity classes apart. Some of these separations are reduced to showing super-polynomial lower bounds for general proof systems. A first step in getting such lower bounds are lower bounds for specific systems.

A general approach to the latter problem that turned out very promising is the following. A so-called Frege proof system consists of a finite set of axiom schemes, and a finite set of rules. Now, if some tautology (as a sequence) has super-polynomial proofs at least one of the axiom schemes, say $\mathcal{S}$ is likely to be included super-polynomially many times in the proofs (possibly in a subsequence, which is not problematic).

Designing a semantics that only fails this axiom scheme $\mathcal{S}$ gives a way to single out those formulas that are not provable without it. With good semantics even counting the minimal number of applications of $\mathcal{S}$ in a proof is possible. This approach has first been applied successfully in [2].

With this approach one can obtain lower bounds if every proof of some tautology requires a large number of applications of $\mathcal{S}$. However, the situation might be more subtle in the following sense. If τ_n is a series of tautologies which does not have polynomial size proofs, it may still be the case that certain proofs π_n of τ_n require super-polynomially many applications of $\mathcal{S}$ whereas other proofs π'_n of τ_n can do with only polynomially many. In the worst case every hard tautology exhibits this behavior with respect to every axiom scheme.

We can summarize the above reasoning in the following easy theorem.

Theorem 2.1. *Let $\mathcal{L}$ be a Frege system of some logic. Let Γ be a subset of axioms of $\mathcal{L}$. Let $\widetilde{\models}$ be a semantical consequence relation such that $\widetilde{\models}$ is closed under all rules of $\mathcal{L}$ and sound for all axioms of $\mathcal{L}$ except for the ones in Γ.*

Let φ be a tautology of $\mathcal{L}$. If for every set Δ of instances of Γ with $|\Delta| \leq n$ we can find a structure M such that

1. *$M \widetilde{\not\models} \varphi$,*
2. *$M \models \Delta$,*

then φ is not provable in $\mathcal{L}$ using less than $(n+1)$ instances from Γ.

Proof. Suppose for a contradiction that there is some proof

$$p \ : \quad p_0, p_1, \ldots, p_i, \ldots, p_m = \varphi$$

in $\mathcal{L}$ with $\leq n$ instances of Γ in p. Let M be a structure validating (in the $\widetilde{\models}$ sense) all these n instances. By induction on i we get that each p_i holds (in the $\widetilde{\models}$ sense) on M which contradicts $M \widetilde{\not\models} \varphi \ (= p_n)$. $\Box$

2.1 Modal logics

As to illustrate the potential of the above mentioned method, let us briefly summarize the content of [2]. For a number of modal logical systems a semantics can be obtained where the distributivity axiom is replaced by the distributivity rule or even a weaker version of it; the so called *transparency rule*:

$$\frac{A \leftrightarrow B}{\Box A \leftrightarrow \Box B}.$$

This semantics allows one to really count the minimal number of applications of the different distributivity axioms needed. For classical tautologies of the form $\alpha(\vec{p}, \vec{r}) \to \beta(\vec{p}, \vec{s})$, where α is monotone in $\vec{p}$, it turns out that there is a very close connection between the minimal number of applications of distributivity in a proof of $\alpha(\vec{\Box p}, \vec{r}) \to \Box\beta(\vec{p}, \vec{s})$ and the number of gates in a monotone circuit interpolating $\alpha(\vec{p}, \vec{r})$ and $\beta(\vec{p}, \vec{s})$.

Having this close connection, it is possible to invoke a result of Razborov's that $\mathsf{Clique}_n^k(\vec{p}, \vec{r}) \to \neg\mathsf{Color}_n^{k-1}(\vec{p}, \vec{s})$ has no polynomial size monotone interpolating circuits to obtain an exponential lower bound.

2.2 Intuitionistic logic

We know that there exist faithful interpretations of propositional intuitionistic logic into modal logics, most prominently $\mathsf{S4}$. The translations that are best known are the following two.

$(p)^{\Box} = \Box p$	$(p)^{\circ} = p$
$(\bot)^{\Box} = \bot$	$(\bot)^{\circ} = \bot$
$(A \wedge B)^{\Box} = A^{\Box} \wedge B^{\Box}$	$(A \wedge B)^{\circ} = A^{\circ} \wedge B^{\circ}$
$(A \vee B)^{\Box} = A^{\Box} \vee B^{\Box}$	$(A \vee B)^{\circ} = \Box(A^{\circ} \vee B^{\circ})$
$(A \to B)^{\Box} = \Box(A^{\Box} \to B^{\Box})$	$(A \to B)^{\circ} = \Box(A^{\circ}) \to B^{\circ}$

However, it seems unlikely that these translations can be used to get lower bound results for intuitionistic logic as a corollary of the lower bounds for modal logics. However, Hrubes has announced a forth-coming paper in which an exponential lower bound for intuitionistic propositional logic is obtained. He uses a different kind of translation and a version of monotone interpolation for intuitionistic logic.

A more direct approach seems also fruitful. That is, again designing a semantics that makes almost all of the axioms true but fails just one or two. For example, we can consider the following set of complete axioms for intuitionistic logic, where $\neg A$ is defined as $A \to \bot$.

1. $A \to (B \to A)$
2. $[A \to (B \to C)] \to [(A \to B) \to (A \to C)]$
3. $A \to A \vee B$, $B \to A \vee B$
4. $(A \to C) \to [(B \to C) \to (A \vee B \to C)]$
5. $A \wedge B \to A$, $A \wedge B \to B$
6. $A \to (B \to A \wedge B)$
7. $\bot \to A$

The only rule involved in this system is MP, Modus Ponens. It does not really matter which Frege system we use as we know from [4] that all Frege systems for intuitionistic logic polynomially simulate each other. We shall now discuss some semantics and their behavior with respect to the axioms.

2.3 A natural candidate

The most natural candidate to focus on when proving a lower bound, seems to be the axiom

$$[A \to (B \to C)] \to [(A \to B) \to (A \to C)].$$

For, if a proof is long, then many parts of the proof have to be combined. If MP is the only rule, the only axiom to combine parts of the proof is indeed this axiom. Moreover, it is known ([5]) that the implicational fragment of intuitionistic logic is already $\mathsf{P}{-}\mathsf{SPACE}$ complete and the above axiom seems to be the only informative axiom in this fragment.

Note the similarity of this axiom with the distributivity axiom in modal logics. In particular, by taking for $A := \top$ and viewing $\top \to \psi$ as $\Box\psi$, we syntactically recognize the distribution axiom. However, this is just a heuristic.

3 Kripke semantics for subintuitionistic logics

A semantics that is sound and complete with respect to all the axioms and rules of intuitionistic propositional logic is the well known Kripke semantics where a model is a triple $\langle W, \leq, V\rangle$. Here W is non-empty set of possible worlds, $\leq$ is a transitive reflexive relation on W, and V is a mapping that tells us which propositional variables hold at which world. In addition V has to be upwards persistent, that is, if $p \in V(a)$ and $a \leq b$, then $p \in V(b)$.

The notion of a formula A being forced in a world a is as usual inductively defined:

$a \Vdash p$	$\Leftrightarrow$	$p \in V(a)$
connectives $\perp$, $\vee$ and $\wedge$:		As usual (commuting)
$a \Vdash A \to B$	$\Leftrightarrow$	$\forall b\ (a \leq b \Vdash A \Rightarrow b \Vdash B)$

The notion of truth is upwards persistent, that is, preserved upwards. The heuristic of this semantics is that $a \leq b$ represents that b is a possible future world of a where possibly more knowledge may have been obtained by the creative subject. We shall refer to this semantics here as "classical" or "usual" Kripke semantics.

From now on we will consider only structures in which the number of possible worlds is **finite**. For classical intuitionistic logic we know that this is no restriction as we have completeness with respect to finite Kripke models.

3.1 General semantical notions

We shall now discuss some variations of Kripke semantics so as to validate almost all axioms but one or two. Again we shall define notions as the forcing relation $\Vdash$ and the like.

With $[A]$ we shall denote the set of worlds where a formula A is forced under some given definition of forcing. With $\overline{[A]}$ we shall denote the smallest set of worlds containing $[A]$ which is closed upwards. Note that in "classical" Kripke semantics we have that $[A] = \overline{[A]}$ for all A.

We say that a forcing relation $\Vdash$ is *upwards persistent* or just *persistent* if $a \Vdash A\ \&\ a \leq b\ \rightarrow\ b \Vdash A$ holds for all A.

3.2 Minimal logic

Minimal logic is as intuitionistic logic, only now omitting the axiom schema $\perp \to A$. It is well known that General Kripke semantics where $\perp$ is considered as a propositional variable is sound and complete for minimal logic.

So, this semantics is an easy example of a semantics that validates all axioms of intuitionistic logic but one. An application of an axiom $\perp \to A$ is sound on a model if $[A] \cap [\perp] = [\perp]$. Looking for a (constructive) tautology for which any proof needs a lot of applications of the axiom $\perp \to A$ is now tantamount to looking for a tautology which has models in minimal logic where it fails to hold, but where for many instances of A, we have that $[A] \cap [\perp] = [\perp]$.

3.3 Long suspense semantics

The long suspense semantics is almost as usual finite Kripke semantics. The only difference is in the truth definition of the implication $\to$. In this semantics we set:

$$a \Vdash A \to B \quad \Leftrightarrow \forall b \ (a \leq b \Vdash A \to \exists c \ b \leq c \Vdash B).$$

The heuristic is as follows. Once the creative subject knows that $A \to B$, in a future world, where he gets to know A he shall obtain B but possibly at some later time as he might need to perform some calculations.

Under this definition of $\Vdash$, it is easy to check by induction on the complexity if a formula that we still have persistency of truth. Note, that $\neg A$, which is short for $A \to \perp$, has the same semantical condition as in classical Kripke semantics.

All axioms remain valid. However, instead of Modus Ponens, we get some weaker versions of it, like:

$$\mathsf{MP}^- : \frac{A \quad A \to B}{\top \to B}; \ \mathsf{MP}_\mathsf{l}^\top : \frac{\top \to A \quad A \to B}{\top \to B}; \ \mathsf{MP}_\mathsf{r}^\top : \frac{\top \to A \quad \top \to (A \to B)}{\top \to B}.$$

Here, $\top$ is a logical constant satisfying the following defining axiom and rule.

$$\top$$
$$\vdash \top \to B \ \Rightarrow \ \vdash \neg\neg B$$

It seems unlikely that this semantics is enough fine-grained to really count the number of applications op Modus Ponens that is needed in a proof of an intuitionistic tautology. In particular, we have that

$$\models_{\text{long suspense}} A \to B \quad \Leftrightarrow \quad \models_i A \to \neg\neg B.$$

3.4 Short suspense semantics

This semantics is as the long suspense semantics, with the only difference that the creative subject is to calculate immediately all consequences of new facts. Let $\leq^\sharp$ be defined as

$$a \leq^\sharp b :\Leftrightarrow (a \leq b \wedge \forall c\ (a \leq c \leq b \rightarrow c = a \vee c = b)).$$

The definition of $\Vdash$ now becomes as follows.

$$a \Vdash A \rightarrow B \quad \Leftrightarrow \forall b\ (a \leq b \Vdash A \Rightarrow \exists c\ (b \leq^\sharp c \Vdash B))$$

Note that the logic corresponding to the short suspense semantics would be different if we allowed infinite structures too.

Again, it is not hard to see that we have persistency of $\Vdash$. And, again, instead of MP we only have MP^-. The only axiom scheme that fails is

$$[A \rightarrow (B \rightarrow C)] \rightarrow [(A \rightarrow B) \rightarrow (A \rightarrow C)].$$

In order to see that this scheme is not valid, we take $A = p$, $B = q$ and $C = r$. Next, we consider the structure

$$a \leq b \leq c \leq d$$

with

$$\begin{array}{rl} V(d) = & \{p, q, r\}; \\ V(c) = & \{p, q\}; \\ V(b) = & \{p\}; \\ V(a) = & \varnothing. \end{array}$$

Clearly, $a \nVdash p \rightarrow r$, as $b \Vdash p$ but for no x with $b \leq^\sharp x$ we have $x \Vdash r$. However, $a \Vdash p \rightarrow q$, so $a \nVdash (p \rightarrow q) \rightarrow (p \rightarrow r)$. On the other hand, $a \Vdash q \rightarrow r$ whence also $a \Vdash p \rightarrow (q \rightarrow r)$. Consequently

$$a \nVdash (p \rightarrow (q \rightarrow r)) \rightarrow ((p \rightarrow q) \rightarrow (p \rightarrow r)).$$

It is very much unclear if this semantics allows one to control the number of applications of $[A \rightarrow (B \rightarrow C)] \rightarrow [(A \rightarrow B) \rightarrow (A \rightarrow C)]$ in an intuitionistic proof.

3.5 Weak persistent semantics

The semantics is as classical Kripke semantics, with the only difference that now we do not demand that V is persistent. Rather, we require V to be *weakly persistent*, that is,

$$a \leq b \rightarrow \exists c{\geq}b \;\; V(a) \subseteq V(c).$$

Note, that if a branch in a model has a top element, this element is maximal along this branch with respect to V. The heuristic is that the creative subject may temporarily forget something, provided that he or she later reconstructs the knowledge at some point.

We now define $\Vdash$ as usual, that is

$$a \Vdash A \rightarrow B \quad \Leftrightarrow \quad \forall b \; (a \leq b \Vdash A \Rightarrow b \Vdash B).$$

We loose upwards persistency of $\Vdash$. Rather, we have that $\Vdash$ is weakly persistent in the sense that $a \Vdash \varphi \;\wedge\; a \leq b \rightarrow \exists c \; b \leq c \Vdash \varphi$. However, $\Vdash$ is strongly persistent for formulas of the form $A \rightarrow B$. The two axioms that fail are

$$A \rightarrow (B \rightarrow A)$$

and

$$A \rightarrow (B \rightarrow (A \wedge B)).$$

However, the weak persistent semantics is sound with respect to the following two rules.

$$\vdash A \quad \Rightarrow \quad \vdash B \rightarrow A$$

and

$$\vdash A \quad \Rightarrow \quad \vdash (B \rightarrow (A \wedge B)).$$

Moreover, we do have restricted versions of the axioms. That is, if A is of the form $E \rightarrow F$, then both axioms are valid indeed. On this semantics MP holds. Note that, e.g., $\not\models (\top \rightarrow A) \leftrightarrow A$. Semantically, axioms of the form $A \rightarrow (B \rightarrow (A \wedge B))$ and of the form $A \rightarrow (B \rightarrow A)$ are equivalent and seem to be interderivable.

However, it seems that all axioms of the form $A \rightarrow (B \rightarrow A \wedge B)$ are as independent as possible. For example

$$\not\models [(p \vee q) \rightarrow (r \rightarrow ((p \vee q) \wedge r))] \rightarrow [p \rightarrow (r \rightarrow (p \wedge r))].$$

As we only consider finite frames, we do have that $A \rightarrow (B \rightarrow \neg\neg(A \wedge B))$ holds. If we allowed also infinite models this could be refuted.

A weak persistent model M being sound with respect to an axiom of the omitted form has a moderately nice characterization.

$$M \models A \to (B \to A) \quad \Leftrightarrow \quad \overline{[A]} \cap [B] \subseteq [A] \qquad (*)$$

Thus clearly, we also have the following sufficient condition.

$$[A] \text{ is upwards closed on } M \quad \Rightarrow \quad M \models A \to (B \to A) \text{ for any } B$$

3.5.1 A linear lower bound

We will use the weak persistent semantics to prove the following lower bound for intuitionistic propositional logic.

Theorem 3.1. *The intuitionistic tautology*

$$p_0 \to (\ldots \to (p_n \to (p_{n+1} \to p_0 \wedge \ldots p_{n+1}))$$

is not provable in intuitionistic propositional logic using less than n schemes *of the form*

$$A_i \to (B \to A_i \wedge B) \quad \textit{for fixed } A_i,$$

or

$$A_i \to (B \to A_i) \quad \textit{for fixed } A_i.$$

From now on, we will consider rooted (weak persistent) models only and denote them $\langle M, r\rangle$ where r is the root.

Definition 3.2. *If $\langle M, r\rangle$ and $\langle M', r'\rangle$ be rooted Kripke structures, e.g., weak persistent models. Let r_0 be a new variable. We define the structure $[\langle M, r\rangle \nwarrow r_0 \nearrow \langle M', r'\rangle]$ by putting $r_0 \le r$, $r_0 \le r'$ and then taking the transitive and reflexive closure.*

Note that if A and B are weak persistent rooted models we can always find $V(r_0)$ such that $[A \nwarrow r_0 \nearrow B]$ becomes a weak persistent rooted model too. This does not hold for classical intuitionistic structures.

Whenever $V(r)$ is defined and $[A \nwarrow r \nearrow B]$ is a well defined model, we shall also use the notation $[A \nwarrow r \nearrow B]$ to refer to that model.

Lemma 3.3. *Let $\langle M, r\rangle$ be a weak persistent rooted model such that $M, r \not\Vdash A$. Let $\langle M', r'\rangle$ be an arbitrary other such model, and choose $V(r_0)$ such that $V(r_0) \subseteq V(r)$. If $[\langle M, r\rangle \nwarrow r_0 \nearrow \langle M', r'\rangle]$ is a weak persistent model, then it holds that*

$$[\langle M, r\rangle \nwarrow r_0 \nearrow \langle M', r'\rangle], r_0 \not\models A.$$

Proof. By an easy induction on the complexity of A. For $A = \bot$ it clearly holds. If A is a propositional variable it holds as $V(r_0) \subseteq V(r)$. The connectives $\wedge$ and $\vee$ are easy. If $A = B \to C$ and $M, r \nVdash B \to C$, we can find $r \leq \tilde{r} \Vdash B$ and $M, \tilde{r} \nVdash C$. But also $r_0 \leq \tilde{r}$ and also

$$[\langle M, r\rangle \nwarrow r_0 \nearrow \langle M', r'\rangle], \tilde{r} \Vdash B \quad \text{and} \quad [\langle M, r\rangle \nwarrow r_0 \nearrow \langle M', r'\rangle], \tilde{r} \nVdash C.$$

$\square$

Let us phrase the following observations in a lemma.

Lemma 3.4. *If $[A]$ is upwards closed on a weak persistent model M, then the following two schemata (as schemata in B) hold on M.*

$$A \to (B \to A \wedge B)$$
$$A \to (B \to A)$$

Proof. By the characterization stated in $(*)$ and the fact that the two schemata are semantically equivalent. $\square$

The proof of Theorem 3.1 now follows from Theorem 2.1 and the following lemma.

Lemma 3.5. *Let the following n axiom schemata be given.*

$$A_i \to (B \to A_i \wedge B) \quad \textit{for fixed } A_i; \quad 1 \leq i \leq n.$$

Let $\vec{q}$ be a string of variables distinct from each of $p_0, \ldots, p_{n+1}$. There exists a model $[\langle M, r\rangle \nwarrow r_0 \nearrow \langle N, r'\rangle]$ where $\vec{q}$ holds at each world, each of the n axiom schemata $A_i \to (B \to A_i \wedge B)$ holds at r_0, however,

$$r_0 \nVdash p_0 \to (\ldots \to (p_n \to (p_{n+1} \to p_0 \wedge \ldots p_{n+1})).$$

Moreover, the $\langle M, r\rangle$ and the $\langle N, r'\rangle$ are classical Kripke models.

Proof. By induction on n. **For $\mathbf{n = 0}$** we consider the simple model consisting of three points $a \leq b \leq c$ where $V(a) = \{p_0, \vec{q}\}$, $V(b) = \{p_1, \vec{q}\}$ and $V(c) = \{p_0, p_1, \vec{q}\}$. Clearly, this is a weak persistent model where $p_0 \to p_1 \to p_0 \wedge p_1$ fails to hold and where, moreover, $\vec{q}$ holds at any world. It is easy to consider this model as $[\varnothing \nwarrow r_0 \nearrow \langle N, r'\rangle]$ with $\langle N, r'\rangle$ a classical Kripke model.

If we now consider $\mathbf{n+1}$ **axiom schemata** $A_i \to (B \to A_i \wedge B)$ for $1 \leq i \leq n+1$, we reason as follows. First we make a case distinction.

If $\nvdash_i \vec{q} \wedge p_0 \to \bigvee_{i=1}^{n+1} A_i$ we find a classical (note that we used $\vdash_i$ in the case distinction) Kripke model $\langle M, r\rangle$ with $r \Vdash p_0, \vec{q}$ but $r \nVdash A_i$ for all i. Next, we consider the Kripke model N consisting of just two points $b \leq c$ with $V(b) = \{p_1, \ldots, p_{n+2}, \vec{q}\}$ and $V(c) = \{p_0, \ldots, p_{n+2}, \vec{q}\}$.

Finally we choose a fresh r_0, define $V(r_0) = \{p_0, \vec{q}\}$ and consider $[\langle M, r\rangle \nwarrow r_0 \nearrow N]$. We now combine Lemma 3.3 and the fact that our two building blocks are classical Kripke models to conclude that the $[A_i]$ are upwards closed on $[M, r \nwarrow r_0 \nearrow N]$. Consequently, by Lemma 3.4 we see that all the axiom schemata $A_i \to (B \to A_i \wedge B)$ hold on this model. Indeed, we also have that $r_0 \nVdash p_0 \to (\ldots \to (p_n \to (p_{n+1} \to p_0 \wedge \ldots p_{n+1}))$.

In case $\vdash_i \vec{q} \wedge p_0 \to \bigvee_{i=1}^{n+1} A_i$, by the well known disjunction/Harrop property (see e.g. [3] or [6] Chapter 3, Corollary 3.1.5) we know that for some A_i we have that $\vdash_i \vec{q} \wedge p_0 \to A_i$. We assume w.l.o.g. that $\vdash_i \vec{q} \wedge p_0 \to A_{n+1}$. We now apply our induction hypothesis to the n axiom schemata $A_i \to (B \to A_i \wedge B)$ for $1 \leq i \leq n$ together with the formula $p_1 \to \ldots p_{n+2} \to p_1 \wedge \ldots \wedge p_{n+2}$. However, we now demand in our call on the induction hypothesis that all of $\vec{q}, p_0$ hold in any point of the model. By Lemma 3.4 it is clear that this model suffices for our purpose.

$\square$

3.6 Quasi filter semantics

This semantics is reminiscent of neighborhood semantics for modal logics. A model is now a quadruple $\langle W, R, V, G\rangle$ such that $\langle W, R, V\rangle$ is a usual Kripke model and G is a set of subsets of W such that $W \in G$. The definition of $\Vdash$ is now only altered for the implication connective. To state this definition, we first need some notation. We will denote by $a\uparrow$ the set $\{b \mid a \leq b\}$. With G_a we mean the set of subsets in G intersected with $a\uparrow$. In a more general approach, the G_a could be defined separately and independent of some overall G.

With $[A]_a$ we mean the set $\{b \in a\uparrow \mid b \Vdash A\}$. Clearly these sets are defined inductively simultaneously with the forcing relation $\Vdash$.

$$a \Vdash A \to B \quad \Leftrightarrow \quad ([A]_a \in G_a \Rightarrow [B]_a \in G_a).$$

Note, that if $[A]_a \in G_a$ and $a \leq b$, then $[A]_b \in G_b$ and hence $\Vdash$ is monotone.

The semantics is closed under MP as $W \in G$. Some axioms require some closure conditions on G. Actually there is a close correspondence.

Axiom scheme	Restriction on G
$A \to (B \to A)$	no restrictions
$(A \to (B \to C)) \to ((A \to B) \to (A \to C))$	?
$A \to A \vee B$, $B \to A \vee B$	G closed under supersets
$(A \to C) \to [(B \to C) \to (A \vee B \to C)]$	?
$A \wedge B \to A, A \wedge B \to B$	G closed under supersets
$A \to (B \to A \wedge B)$	G closed under intersections
$\perp \to A$	$\emptyset \notin G$

This semantics seems rather promising with respect to direct proofs in lower bounds for two reasons. First, by altering properties of G we can make some axioms true and others not. Second, this approach is similar to the approach that was applied to modal logics in [2] and proved fruitful. In particular, one can say what it means for a certain instance of the axiom to hold on the model and hence the number of axioms can be controlled. A significant difference is with the finite model property. Given a finite number of variables, the canonical model in many modal logics is finite. However, this is not the case in intuitionistic logic.

3.7 Basic Propositional Logic

A logic that has been studied in the literature (e.g., see [1]) is so-called Basic Propositional Logic (sometimes also called Visser's logic). It turns out that this logic corresponds to regular Kripke semantics where the underlying Kripke frames are not necessarily reflexive and where the meaning of $\to$ is defined as follows.

$$a \Vdash A \to B \quad \Leftrightarrow \quad \forall b\ ((a < b) \wedge b \Vdash A \ \Rightarrow b \Vdash B)$$

4 Other logics

The semantic approach to obtain lower bounds can be applied to any other logic, in particular to classical propositional logic. In the case of classical logic, we are interested in semantics that fail only some of the axioms. Classical logic is obtained by adding to the axioms of intuitionism the axiom schema

$$\neg\neg A \to A.$$

Any known sub-logic of classical logic with a good semantics can serve as a candidate to prove lower bounds for classical logic.

4.1 Intuitionistic logic

We could hope that some tautologies require a large number of axioms of the form $\neg\neg A \to A$. However, this hope is in vain as we have Glivenko's theorem which says that for propositional logical A we have:

$$\vdash_c A \leftrightarrow \vdash_i \neg\neg A.$$

Thus, every classical tautology has a proof with just one application of excluded middle of the form $\neg\neg A \to A$.

4.2 Hybrid logics

It is also possible to consider a logic where both $\neg$ and $\to$ are primitive symbols where the $\neg$ is defined classically and the $\to$ intuitionistically. Clearly, in this case we have $p \vee \neg p$ and $\neg p \not\equiv (p \to \bot)$. The relation of this logic to $\mathsf{S4}$ is evident as $\Box A \wedge A$ can be defined as $\top \to A$.

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