The Logic of Provability

Suggestions for exercises. Week 2.

1. (C)

   Provide the code of $\bot \to \bot$. If we were to code $\neg(v_0 = v_1)$, how should we treat the brackets?

2. (C)

   What is the least common multiple of 3 and 5? Find x smaller than this least common multiple such that $x \equiv 2(3)$ and $x \equiv 3(5)$.

3. (B)

   Prove (for example by induction) that $\Sigma_{k=0}^{n} k = \frac{n(n+1)}{2}$.

4. (C)

   Shoenfield's coding is given by $\pi(x, y) = (x + y)(x + y) + x + 1$. Determine the least positive natural number that is not in the range of this function. (For example by just calculating the "first values".) (($*$) Determine the lengths of consecutive gaps in the range of this pairing function.)

5. (C)(D)

   When using real numbers we can write down a function for the second component of a pair, so, if $z = (x, y)$ then $y = \frac{z}{2} - (\lfloor\sqrt{\frac{z}{2}}\rfloor^2 + 1)$. Show this and give a similar function for x.

6. (C)(D)

   Does the number 356645864186 code a formula? If so, which formula?

7. (D)

   Find $x < \text{lcm}(3, 4, 7)$ such that $x \equiv 2(3)$, $x \equiv 1(4)$ $x \equiv 5(7)$

8. (C) (A)

   Prove (41) on page 36 of chapter 2. Prove the claim that $\text{GN}(t) < \text{GN}(t = t')$. Prove also that $\text{GN}(t') < \text{GN}(t)$ whenever t' is a proper subterm of t.

9. (C)

   Prove that

   $$\mathbf{GL} \vdash \bot \quad \Rightarrow \quad \mathbf{GL} \vdash A$$

   for any formula A.

10. (C)

    Prove that if $\mathbf{K} \vdash A \to B$ and $\mathbf{K} \vdash A \to C$ then $\mathbf{K} \vdash A \to (B \wedge C)$.

11. (C)

    Formulate the axiom schemas $\Box A \to \Box\Box A$ and $\Box(\Box A \to A) \to \Box A$ in terms of the $\Diamond$ modality.

12. (C)

    Derive the following formulas in the respective logics:

    (a) $\mathbf{K} \vdash \Box(\varphi \wedge \psi) \to \Box\varphi$
    (b) $\mathbf{K} \vdash (\Box\varphi \wedge \Box\psi) \to \Box(\varphi \wedge \psi)$
    (c) $\mathbf{K} \vdash (\Box\varphi \vee \Box\psi) \to \Box(\varphi \vee \psi)$
    (d) $\mathbf{K4} \vdash \Box\varphi \to \Box\Box\Box\varphi$
    (e) $\mathbf{K4} \vdash (\Box\varphi \wedge \Box\Box\psi) \to \Box\Box(\varphi \wedge \psi)$
    (f) $\mathbf{S4} \vdash \Diamond\varphi \leftrightarrow \Diamond\Diamond\varphi$
    (g) $\mathbf{K} \vdash \Box p \wedge \neg\Box\bot \to \neg\Box\neg p$
    (h) $\mathbf{K4} \vdash \Box p \wedge \Box\Box(p \to q) \to \Box\Box q$
    (i) $\mathbf{K} \vdash \Box\Box p \wedge \Box\Box(p \to q) \to \Box\Box q$

13. (D)

    Recast the inductive definition of a modal sentence as an explicit one employing finite sequences.

14. (C)

    Fill in the gaps in theorem 9 of chapter 1.

15. (C)

    Prove a **K4** variant of Theorem 20 of Chapter 1 of the book.
    $\mathbf{K4} \vdash \boxdot A_1 \wedge \ldots \boxdot A_n \to B \Rightarrow \mathbf{K4} \vdash \Box A_1 \wedge \ldots \Box A_n \to \Box B$

16. (D)

    Complete the proofs of theorems 14 and 15 of chapter 1.

17. (C) (A)

    Prove that $\mathbf{K} \vdash \Diamond(A \vee B) \leftrightarrow \Diamond A \vee \Diamond B$

18. (C) (A)

    Prove that $\mathbf{K} \vdash \Box A \to \Box(\Box A \to A)$ and
    $\mathbf{K} \vdash \Box(\Box p \to p) \to (\Box(\Box q \to \Box p) \to \Box(\Box q \to p))$.

19. (D)(*)

    Define formally a natural deduction system for **GL**. We thus would like to have the necessitation rule only applicable in case of no open assumptions. Prove equivalence of the deduction formulation and the formulation in the book.