The Logic of Provability
Suggestions for exercises. Week 4.

1. (C)

   Find realizations $*$ and $\sharp$ such that

   - $\mathsf{PA} \vdash (\Box p)^*$
   - $\mathsf{PA} \nvdash (\Box p)^\sharp$

2. (C) (A)

   Consider the following sentences: $\bot$, $0 = S0$, and $\bot \to \bot$. Also consider the following formulas $x = 3$, $x = 1$ and $x = 458$. Which sentence is a fixed point of which formula? Do we have uniqueness of fixed points? (Modulo provable equivalence?) ((*) Which fixed points would have been found if the construction from the fixed point theorem of chapter 3 would have been applied? The (*) part of this exercise does not belong to the obligatory homework.)

3. (D)

   Suppose there were a predicate $\mathrm{Tr}(x)$ that holds precisely on the codes of true (in the standard model) sentences. Use the fixed-point theorem to formalize the liar paradox and to conclude that our assumption is wrong. We have thus proved Tarski's theorem on the non-definability of truth. There does exist however a formula $\mathrm{Tr}_{\Sigma_1}(x)$ that holds only on codes of true (in the standard model) Σ_1 sentences. Why is our previous argument not applicable to the $\mathrm{Tr}_{\Sigma_1}(x)$ predicate? ((*) Use the above argumentation to show that all the inclusions in the arithmetic hierarchy are strict.)

4. (C)

   Show that $\mathbf{GL} \vdash \neg\Box\Box\bot \to (\neg\Box\neg\Box\bot \wedge \neg\Box\neg\neg\Box\bot)$. What is the arithmetical content of this formula?

5. (C)

   Show that $\mathbf{GL} \vdash \Box((\Box p \to p) \to \neg\Box\Box\bot) \to \Box\Box\bot$.

6. (C)

   Provide a proof in $\mathbf{GL}$ of both $\Box\bot \to \Box\Diamond p$ and $\Box\Diamond p \to \Box\bot$.

7. (D)

   Show that **K4** plus Löb's rule is the same logic as **GL**. Löb's rule allows us to conclude $\Box A$ whenever we have a derivation of $\Box(\Box A \to A)$.

8. (C)

   Show that $\mathbf{K4} \vdash \Box A \to \Box(\Box A \wedge A)$

9. (C) (A)

   Show that $\mathbf{K} \vdash \Box A \to \Box(\Box\Box A \wedge \Box A \to \Box A \wedge A)$ and also that $\mathbf{K} \vdash \Box A \to \Box(\Box(\Box A \wedge A) \to \Box A \wedge A)$. Show that $\mathbf{K} \vdash \Box(\Box A \wedge A) \to \Box\Box A$. Finally show that $\mathbf{GL} \vdash \Box A \to \Box\Box A$. Prove that $\mathbf{K4} \subset \mathbf{GL}$.

10. (C) (A)

    Prove that $\mathbf{GL} \vdash \Diamond A \wedge \Box B \to \Diamond(A \wedge B)$.
    Prove that $\mathbf{GL} \nvdash \Diamond A \wedge \Box B \to \Diamond\Diamond(A \wedge B)$.
    Prove also that $\mathbf{GL} \nvdash \Box\Box A \wedge \Diamond B \to \Diamond(A \wedge B)$.

11. (C)

    Prove that $\mathbf{K4} \nvdash \Box(\Box A \to A) \to \Box A$. Is it possible to find a finite countermodel?

12. (D)

    Clonnectives (forget about this term after this exercise (Lev says they are just called connectives)) comprise the following symbols: $\{\neg, \Box, \Diamond, \to, \wedge, \vee\}$. If a modal sentence contains n clonnectives, how many subsentences does it maximally have? Give an example where this maximum is met and give an example where this maximum is not met. So, how many "formulas" (i.e. subsentences and negated subsentences) does it maximally have? How many maximal proto-consistent sets of formulas does it have? We call a set X proto-consistent if $A \in X \Rightarrow \neg A \notin X$ and $\neg A \in X \Rightarrow A \notin X$. Give an example of a proto-consistent set in **K** that is not consistent in **K**.

13. (C)

    Write down all the formulas (i.e. subsentences and negated subsentences) of $\Box p \to p$. Give all the maximal **K4**-consistent sets corresponding to $\Box p \to p$. Indicate which sets are related to each other in the sense of the **K4** relation defined on page 80 of the book and thus indicate what a **K4** countermodel of $\Box p \to p$ can look like.

14. (C)

    (a) Give an example of a **K4**-consistent formula which is not **S4**-consistent. (b) The same question for the logics **K** and **K4**.

15. (C)

    Show: $\Box(\Box p \to p) \to \Box p$ is true in all upwards well-founded[1], transitive Kripke models.

16. (C)

    Let M be a Kripke model and $x \in M$. Show: the set $\{\phi : (M, x) \models \phi\}$ is maximal consistent.

17. (C)

    Show that any maximal consistent set of formulas is closed under modus ponens.

[1] That is, there is no infinite chain $x_1 R x_2 R x_3 R \ldots$ of elements of the model.