The Logic of Provability
Suggestions for exercises. Week 6.

1. (C)

   We have the following conditionals to be provable in the indicated logics.

   (a) $\mathbf{K} \vdash \Box p \wedge \neg\Box\bot \to \neg\Box\neg p$
   (b) $\mathbf{K4} \vdash \Box p \wedge \Box\Box(p \to q) \to \Box\Box q$
   (c) $\mathbf{K} \vdash \Box\Box p \wedge \Box\Box(p \to q) \to \Box\Box q$

   Show for each of this implications that the direction can not be reversed, that is, we have no equivalences in the logics.

2. (C) (D) (*)

   Show that **K** plus Löb's rule proves precisely the same theorems as **K**. This is rather surprising since we know that **K4** plus Löb's rule is extensionally the same as **GL** which properly extends **K4**.

3. (C) (A)

   Prove that $\mathbf{GL} \vdash \Box A \quad \Rightarrow \mathbf{GL} \vdash A$.

4. (C) (D)

   Prove that T is modally complete with respect to the class of reflexive frames. (Hint: we only need to alter the definition of the accessability relation R from the completeness proof of **K** a bit.)

5. (C) (D)

   Prove that $\mathsf{S4}$ is modally complete with respect to the class of reflexive and transitive frames.

6. (C)

   Let *Blöb's* rule be the rule that from $\Box(\Box A \to A)$ we may conclude $\Box A$. Show that the logic consisting of **K4** plus Blöb's rule proves precisely the same theorems as **GL**.

7. (C) (D)

   Using the completeness theorem for **S4** prove:

   $$\mathbf{S4} \vdash \Box\varphi \vee \Box\psi \text{ implies } \mathbf{S4} \vdash \varphi \text{ or } \mathbf{S4} \vdash \psi.$$

   (Hint: a countermodel for $\Box\varphi \vee \Box\psi$ can be combined from those of φ and ψ.)

8. (C) (D)

   Describe a class of Kripke models under which the logic **K** together with the axiom $\Box\Box\bot$ is sound and complete.

9. (C)

   For this exercise our language will be $\{+, \cdot, 0, \mathsf{S}, \mathsf{lh}(\text{-}), (\text{-})_x\}$. Here $\mathsf{lh}(\text{-})$ is the function that assigns to each finite sequence the length of the sequence. $(\text{-})_x$ gives the x-th element of the sequence. So if s is the code of the sequence $[12, 15, 8, 31]$ we have $\mathsf{lh}(s) = 4$, and for example $(s)_3 = 8$. Further we may assume the existence of a Σ predicate $\mathsf{Finseq}(x)$ that holds only on codes of finite sequences. Give formulas G_i such that

   - $G_0(a, b) \Leftrightarrow \mathbb{N} \models 2^a = b$
   - $G_1(a, b) \Leftrightarrow \mathbb{N} \models a! = b$
   - $G_2(a, b) \Leftrightarrow \mathbb{N} \models 2^a + a = b$
   - $G_3(a, b) \Leftrightarrow \mathbb{N} \models 2^a + a! = b$
   - $G_4(a, b) \Leftrightarrow \mathbb{N} \models a^a = b$
   - $G_5(a, b) \Leftrightarrow \mathbb{N} \models a^{a^a} = b$

   Plea that the same can be done without adding $\mathsf{lh}(\text{-})$ and $(\text{-})_x$ to our language.

10. (D)

    The set of polynomials P in the variable x is defined as $\{\Sigma_{i=0}^{\infty} a_1 x^i \mid a_i \in \mathbb{N} \quad a_i = 0 \text{ except for finitely many } i\}$. We leave out all the contributions with coefficient $a_i = 0$. Prove that any term in one variable in the language of PA is functionally equivalent to a polynomial, that is, they are the same function. Conclude that 2^x can never be given as a term of PA.

11. (C) (A)

    Make a rectangle whose cornerpoints consist of the following statements: $\mathbb{N} \models \varphi$, $\mathbb{N} \models \mathsf{Bew}(\ulcorner\varphi\urcorner)$, $\mathsf{PA} \vdash \varphi$ and $\mathsf{PA} \vdash \mathsf{Bew}(\ulcorner\varphi\urcorner)$. Indicate which statement implies which by means of (labeled) arrows. Give account for the implications.

12. (C)

    Let F_m, the formula with Gödel number m be as defined on page 127. Give F_1 and F_2.

13. (C) (A)

    Show that $\mathrm{Pf}(x, \ulcorner\neg S_i\urcorner) \wedge \mathrm{Pf}(x, \ulcorner\neg S_j\urcorner)$ is impossible if $i \neq j$, where the S_i are defined as on the final line of page 127. Prove by induction on a that $H(a, b)$ as defined on page 128 indeed, verifiably in PA, defines a function, that is, $\mathsf{PA} \vdash \forall a \forall b \forall c\ (H(a, b) \wedge H(a, c) \to b = c)$.

14. (C)

    The first step in the Solovay proof is extending our countermodel. Do we have that $0 \Vdash A \leftrightarrow 1 \Vdash A$ for all modal formulas A? If so, provide a proof, if not a counterexample.

15. (C)

    The graph $H(a, b)$ of the Solovay function is defined on page 127 by an application of the generalized diagonal lemma of page 53. What are m and n in this application?

16. (C)

    Prove that $\mathsf{PA} \vdash H(a, \bar{i}) \to S_i$ whenever i is a topnode in our model.

17. (C)

    Show that $i \geq 1 \;\Rightarrow\; \mathsf{PA} \vdash H(a, \bar{i}) \to \mathsf{Bew}(\ulcorner\neg S_i\urcorner)$.

18. (C) (D)

    What is the logical complexity of the Solovay sentences S_i?

19. (C)

    Consider the proof of Lemma 1 of chapter 9 on page 130. Where is the fact used that B is a subsentence of $\neg A$?